Math 221 - Section 5

A quick guide to homogeneous linear systems of DEs and the phase plane

A *standard form homogeneous* linear system of two first order differential equations is

$$\frac{dx}{dt} = ax + by \qquad\qquad \frac{dy}{dt} = cx + dy$$

where a, b, c, and d are constants.

Homogeneous systems always have a critical point at $(0,0)$; that is, the constant functions $x(t) \equiv 0$ and $y(t) \equiv 0$ are solutions of this system for any values of a, b, c, and d.

To solve the system when $b \neq 0$, we solve the first DE for y and using this to eliminate the variable y from the second DE:

$$\boxed{y = \frac{1}{b}(x' - ax)} \qquad\qquad \boxed{x'' - (a+d)x' + (ad - bc)x = 0\ .}$$

The behavior around the critical point (0,0) depends on the characteristic roots of the auxiliary equation. For example, the critical point can be stable, unstable (a source), or asymptotically stable (a sink). The critical point is described in various ways as a node, saddle, spiral, center, or star. See the pictures on the last page of this handout for pictorial definitions of these terms. **Exercise (1):** In Example (A) of the Solving linear systems handout we found that the general solution to the linear system in $\frac{dx}{dt} = x+3y$, $\frac{dy}{dt} = 0x+2y$ is

$x(t) = 3C_1e^{2t} + C_2e^t \qquad\qquad y(t) = C_1e^{2t}$.

Graph the two solutions where $C_1 = 0$ and $C_2 = 1$ and where $C_1 = -1$ and $C_2 = 0$ in the phase plane. What does this tell you about $(0,0)$? **Example (B):** In Example (B) of the Solving linear systems handout we found that the general solution to the linear system $\frac{dx}{dt} = x + 2y$, $\frac{dy}{dt} = -5x - y$ is

$x(t) = C_1\sin(3t) + C_2\cos(3t)$, $\quad y(t) = C_1\left(\frac{3}{2}\cos(3t) - \frac{1}{2}\sin(3t)\right) + C_2\left(-\frac{1}{2}\cos(3t) - \frac{3}{2}\sin(3t)\right)$.

This general solution involves only cosine and sine functions, but not exponential functions, so $x(t)$ and $y(t)$ can't (asymptotically) approach 0, and they also can't go to $\pm\infty$. The critical point (0,0) for this linear system is a stable center. **Example (C-D):** In Examples (C-D) of the Solving linear systems handout we found that the general solution to the linear system $\frac{dx}{dt} = -\frac{1}{3}x + \frac{1}{12}y$, $\frac{dy}{dt} = \frac{1}{3}x - \frac{1}{3}y$ is

$x(t) = C_1e^{-1/6)t} + C_2e^{-(1/2)t} \qquad\qquad y(t) = 2C_1e^{-(1/6)t} - 2C_2e^{-(1/2)t}$.

As $t \to \infty$, the functions $x(t)$ and $y(t)$ in these solutions (asymptotically) approach 0. The critical point (0,0) for this linear system is a node sink. **Exercises:** For the following systems, find the general solution and discuss the stability of the critical point $(0,0)$: (2) $\dfrac{dx}{dt} = 4x$, $\dfrac{dy}{dt} = x + 4y$. (3) $\dfrac{dx}{dt} = 4x - 5y$, $\dfrac{dy}{dt} = 2x + 6y$.