**Math 221 - Section 5**

# A quick guide to sketching phase planes

Our text discusses equilibrium points and analysis of the phase plane. However, there is one idea, not mentioned in the book, that is very useful to sketching and analyzing phase planes, namely *nullclines*. Recall the basic setup for an autonomous system of two DEs:

$$\frac{dx}{dt} = f(x,y)$$
$$\frac{dy}{dt} = g(x,y)$$

To sketch the phase plane of such a system, at each point $(x_0, y_0)$ in the $xy$-plane, we draw a vector starting at $(x_0, y_0)$ in the direction $f(x_0, y_0)\mathbf{i} + g(x_0, y_0)\mathbf{j}$.

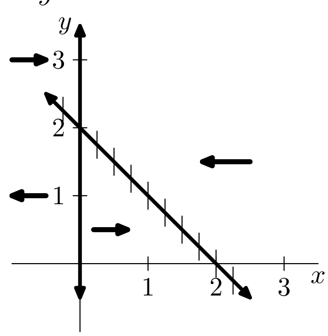
**Definition of nullcline.** The $x$-**nullcline** is a set of points in the phase plane so that $\frac{dx}{dt} = 0$. Geometrically, these are the points where the vectors are either straight up or straight down. Algebraically, we find the $x$-nullcline by solving $f(x,y) = 0$.

The $y$-**nullcline** is a set of points in the phase plane so that $\frac{dy}{dt} = 0$. Geometrically, these are the points where the vectors are horizontal, going either to the left or to the right. Algebraically, we find the $y$-nullcline by solving $g(x,y) = 0$.

**How to use nullclines.** Consider the system

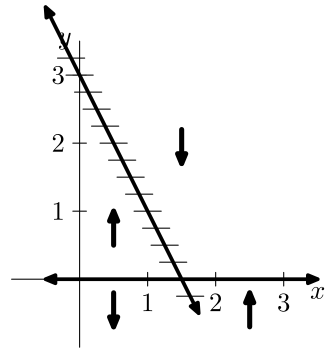
$$\frac{dx}{dt} = 2x\left(1 - \frac{x}{2}\right) - xy,$$
$$\frac{dy}{dt} = 3y\left(1 - \frac{y}{3}\right) - 2xy.$$

To find the $x$-nullcline, we solve $2x\left(1 - \frac{x}{2}\right) - xy = 0$, where multiplying out and collecting the common factor of $x$ gives $x(2 - x - y) = 0$. This gives two $x$-nullclines, the line $x + y = 2$ and the $y$-axis. Solutions of this system move to the right if $x(2 - x - y) > 0$, so the direction field arrows will point toward the right (and either upward or downward) in two cases: When both $x > 0$ and $(2 - x - y) > 0$ (to the right of the $y$ axis and below the line $y = -x + 2$), and when both $x < 0$ and $(2 - x - y) < 0$ (to the left of the $y$ axis and above the line $y = -x + 2$).

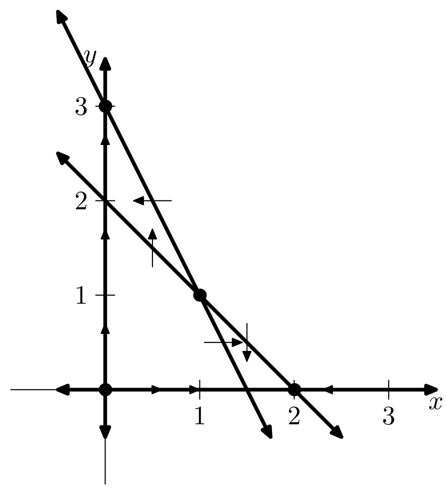



To find the $y$-nullcline, we solve $3y\left(1 - \frac{y}{3}\right) - 2xy = 0$, where multiplying out and collecting the common factor of $y$ gives $y(3 - y - 2x) = 0$. This gives two $y$-nullclines, the line $2x + y = 3$ and the $x$-axis. Solutions of this system move upward if $y(3 - y - 2x) > 0$, so the direction field arrows will point up (and either right or left) in two cases: When both $y > 0$ and $(3 - y - 2x) > 0$

(above the $x$ axis and below the line $y = -2x + 3$), and when both $y < 0$ and $(3 - y - 2x) < 0$ (below the $x$ axis and above the line $y = -2x + 3$).




Combining this information gives us the following picture. Notice that we can draw directions on each nullcline by using the direction information from the other graph. For example, the line segment from $(1, 1)$ to $(0, 3)$, since it is above the line $y = -x + 2$ and to the right of the $y$-axis, has solutions moving to the left.




Also, where the $x$-nullcline and $y$-nullcline cross, both $\dfrac{dx}{dt}$ and $\dfrac{dy}{dt}$ are zero. So these points (marked by dots in the above graph) are equilibrium points.

Once a solution enters the triangle with vertices $(1, 1)$, $(0, 2)$ and $(0, 3)$, it can never leave. Similarly, solutions in the triangle with vertices $(1, 1)$, $(3/2, 0)$ and $(2, 0)$ can never leave.

**Exercises.** Graph the nullclines, sketch the direction fields, and discuss the possible fates of solutions for the following systems. Note: The nullclines may not be straight lines.

(1) $\dfrac{dx}{dt} = x(-x - 3y + 150), \quad \dfrac{dy}{dt} = y(-2x - y + 100).$

(2) $\dfrac{dx}{dt} = x(10 - x - y), \quad \dfrac{dy}{dt} = y(30 - 2x - y).$

(3) $\dfrac{dx}{dt} = 2x\left(1 - \dfrac{x}{2}\right) - xy, \quad \dfrac{dy}{dt} = y\left(\dfrac{9}{4} - y^2\right) - x^2y.$

(4) $\dfrac{dx}{dt} = x(-4x - y + 160), \quad \dfrac{dy}{dt} = y(-x^2 - y^2 + 2500).$

## Math 221 - Section 5

## A quick guide to homogeneous linear systems of DEs

A *standard form* homogeneous linear system of two first order constant coefficient differential equations is

$$\frac{dx}{dt} = ax + by \qquad\qquad \frac{dy}{dt} = cx + dy$$

where $a$, $b$, $c$, and $d$ are constants. Both $x$ and $y$ are dependent variables, depending on the independent variable $t$. The *solution* to this system is a pair of functions $x(t)$ and $y(t)$ that satisfy both DEs together. Often, it will be convenient to write the solution as a column vector

$$\vec{s}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} .$$

To solve this system when $b \neq 0$, we use the first DE to *eliminate* the variable $y$ from the second DE. Solving the first DE for $y$:

$$\boxed{y = \frac{1}{b}(x' - ax)\ .}$$

Then the derivative

$$y' = \frac{1}{b}x'' - \frac{a}{b}x'\ .$$

Plugging these into the second DE of the system gives

$$\frac{1}{b}x'' - \frac{a}{b}x' = cx + d(\frac{1}{b}(x' - ax))\ .$$

Rearranging the terms of this equation and multiplying both sides by $b$, we now have one second order linear constant coefficient DE to solve for $x$, namely

$$\boxed{x'' - (a+d)x' + (ad - bc)x = 0\ .}$$

So we solve this DE in the usual way (chap. 3 sec. 3 of the text). Once we have $x$, we can use the formula in the first box to find $y$.

Using the vocabulary discussed in chapter 5 of the text, the characteristic roots $r_1, r_2$ from the auxiliary equation for the constant coefficient second order homogeneous DE above for $x$ are also the *eigenvalues of the matrix*

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} .$$

An *eigenvector* associated to the root $r_i$ is the vector $\begin{bmatrix} 1 \\ \frac{r_i - a}{b} \end{bmatrix}$. These vectors are discussed further in examples below.

Homogeneous linear systems always have a critical point at $(0, 0)$; that is, the constant functions $x(t) \equiv 0$ and $y(t) \equiv 0$ are solutions of this system for any values of $a$, $b$, $c$, and $d$. The behavior around the critical point (0,0) depends on the characteristic roots of the auxiliary equation. For example, the critical point can be stable, unstable (a source), or asymptotically stable (a sink). The critical point is described in various ways as a node, saddle, spiral, center, or star. See the pictures on the last page of this handout for pictorial definitions of these terms.

**Example (A):** Find the general solution to the linear system

$$\frac{dx}{dt} = 0x - 2y\ , \qquad \frac{dy}{dt} = x + 3y\ ,$$

and analyze the phase plane direction field for this system.

*Answer*: Here we have $b = -2 \neq 0$ in the first DE, we can solve for $y$ and get $y = -\frac{1}{2}x'$. Plugging this into the second DE, then $-\frac{1}{2}x'' = x + 3(-\frac{1}{2}x')$. Then the associated second order equation (also from the boxed formula on p. 1) is $x'' - 3x' + 2x = 0$. This has roots $1, 2$, so the solution for $x$ is

$$x(t) = C_1e^t + C_2e^{2t}.$$

Using the formula (above or from the boxes) for $y(t)$, we get $y = -\frac{1}{2}x' = -\frac{1}{2}C_1e^t - C_2e^{2t}$.

The general solution for the system is

$$x(t) = C_1e^t + C_2e^{2t} \qquad\qquad y(t) = -\frac{1}{2}C_1e^t - C_2e^{2t} \ .$$

Written as a vector, the general solution is $\vec{s}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1e^t \begin{bmatrix} 1 \\ -\frac{1}{2} \end{bmatrix} + C_2e^{2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

(*Note :* It is important to notice that the same constants appear in the solutions for both $x(t)$ and $y(t)$, so we cannot replace a multiple of an arbitrary constant with an arbitrary constant in only one of the solutions.)

The $x$-nullcline is the line $y = 0$; arrows point to the right when $y < 0$ and to the left when $y > 0$. The $y$-nullcline is the line $y = -\frac{1}{3}x$; arrows point upward when $y > -\frac{1}{3}x$ and downward when $y < -\frac{1}{3}x$.

The solution with $C_1 = 1$ and $C_2 = 0$ is given by $x(t) = e^t, y(t) = -\frac{1}{2}e^t$; then for all $t$, this solution follows the ray $y = -\frac{1}{2}x$ for $x > 0$ and $y < 0$ in the phase plane, going up and to the right in the direction of the eigenvector $\begin{bmatrix} 1 \\ -\frac{1}{2} \end{bmatrix}$ for the eigenvalue/root $r_1 = 1$ as $t$ increases. The solution with $C_1 = 0$ and $C_2 = 1$ has $x(t) = e^{2t}, y(t) = -e^{2t}$, and so follows the ray $y = -x$ with $x > 0$ and $y < 0$ in the direction of the eigenvector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ for the eigenvalue/root $r_2 = 2$ as $t \to \infty$.

For the solution of the IVP with $x(0) = 12$ and $y(0) = -5$, as $t \to \infty$ we have $x(t) \to -\infty$ and $y(t) \to \infty$. However, for this IVP the solution $(x(t), y(t))$ can never reach the point $(-7, 6)$.

The critical point $(0, 0)$ is unstable; the characteristic roots 1,2 for $x$ distinct positive real numbers, and so the critical point is a node source.

**Example (B):** Find the general solution of the system

$$\frac{dx}{dt} = x + 2y \ , \qquad \frac{dy}{dt} = -5x - y \ ,$$

and analyze stability of the critical point (0,0).

*Answer*: We have $b = 2 \neq 0$ in the first DE, we can solve for $y$ and get $y = \frac{1}{2}x' - \frac{1}{2}x$. Plugging this into the second DE, then $\frac{1}{2}x'' - \frac{1}{2}x' = -5x - (\frac{1}{2}x' - \frac{1}{2}x)$. Then the associated second order equation is $x'' + 9x = 0$. This has roots $\pm 3i$, so the solution for $x$ is

$$x(t) = C_1 \sin(3t) + C_2 \cos(3t).$$

Using the formula for $y(t)$, we get

$$\begin{aligned} y(t) &= (1/2)(3C_1 \cos(3t) - 3C_2 \sin(3t)) - (1/2)(C_1 \sin(3t) + C_2 \cos(3t)) \\ &= C_1 \left((3/2)\cos(3t) - (1/2)\sin(3t)\right) + C_2 \left(-(1/2)\cos(3t) - (3/2)\sin(3t)\right). \end{aligned}$$

Written as a vector, the general solution is

$$\vec{s}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 \begin{bmatrix} \sin(3t) \\ (3/2)\cos(3t) - (1/2)\sin(3t) \end{bmatrix} + C_2 \begin{bmatrix} \cos(3t) \\ -(1/2)\cos(3t) - (3/2)\sin(3t) \end{bmatrix}.$$

This general solution involves only cosine and sine functions, but not exponential functions, so $x(t)$ and $y(t)$ can't (asymptotically) approach 0, and they also can't go to $\pm\infty$. The critical point (0,0) for this linear system is a stable center.

**Example (C):** Two large tanks are interconnected by pipes. Fresh water flows into tank A at a rate of 6 L/min. Solution flows through a pipe from tank A to tank B at a rate of 8 L/min, and solution flows through another pipe from tank B to tank A at a rate of 2 L/min. The solution in tank B also drains out onto the ground at a rate of 6 L/min. The solutions in both tanks are kept evenly mixed. Initially tank A contains 3 g of salt dissolved in 24 L of water, and tank B contains 5 g of salt dissolved in 24 L of water. Set up an initial value problem modeling this situation.

*Answer:* Let $t$ be the time in minutes, with initial time $t = 0$. Let $x(t)$ be the mass of salt in grams in tank A at time $t$, and let $y(t)$ be the mass of salt in grams in tank B at time $t$. The initial conditions say that $x(0) = 3$ and $y(0) = 5$.

The volume of solution in tank A is changing at a rate of $6 - 8 + 2 = 0$ L/min, so the volume in that tank is a constant 24 L. The volume in tank B changes at a rate of $8 - 2 - 6 = 0$ L/min, so the volume in that tank is also a constant 24 L/min.

First analyze the input and output of salt at tank A. The fresh water flowing into tank A contains 0 g/L of water, so the input from this pipe changes the mass of salt in tank A at a rate of (0 g/L)(6 L/min)=0 g/min. The solution flowing from tank A to tank B removes salt from tank A at a rate of ($\frac{x(t)}{24}$ g/L)(8 L/min). The solution flowing from tank B to tank A inputs salt into tank A at a rate of ($\frac{y(t)}{24}$ g/L)(2 L/min). Since $\frac{dx}{dt}$ = (input rate) - (output rate), then $\frac{dx}{dt} = 0 + \frac{2y}{24} - \frac{8x}{24}$ g/min.

Next do the same for tank B: $\frac{dy}{dt}$ = (input rate) - (output rate) = $\frac{8x}{24} - \frac{2y}{24} - \frac{6y}{24}$ g/min.

The initial value problem modeling this situation is:

$$\frac{dx}{dt} = -\frac{1}{3}x + \frac{1}{12}y\ , \qquad \frac{dy}{dt} = \frac{1}{3}x - \frac{1}{3}y\ ; \qquad x(0) = 3\ , \qquad y(0) = 5\ .$$

**Example (D):** Solve the initial value problem from Example (C), and discuss what happens to the solution as $t \to \infty$.

*Answer.* Here we have $a = -\frac{1}{3}$, $b = \frac{1}{12}$, $c = \frac{1}{3}$, and $d = -\frac{1}{3}$, with $b \neq 0$. The associated second order DE is $x'' - (-\frac{1}{3} - \frac{1}{3})x' + ((-\frac{1}{3})(-\frac{1}{3}) - (\frac{1}{12})(\frac{1}{3}))x = 0$, or $x'' + \frac{2}{3}x' + \frac{1}{12}x = 0$. This DE has auxiliary equation $r^2 + \frac{2}{3}r + \frac{1}{12} = 0$, with roots $r_1 = -\frac{1}{6}$ and $r_2 = -\frac{1}{2}$. Then the solution for $x$ is

$$x(t) = C_1 e^{-(1/6)t} + C_2 e^{-(1/2)t}.$$

Then

$$\begin{aligned} y &= (1/\tfrac{1}{12})(x' - (-\tfrac{1}{3})x) = 12x' + 4x \\ &= 12(-\tfrac{1}{6}C_1 e^{-(1/6)t} - \tfrac{1}{2}C_2 e^{-(1/2)t}) + 4(C_1 e^{-(1/6)t} + C_2 e^{-(1/2)t}) \\ &= 2C_1 e^{-(1/6)t} - 2C_2 e^{-(1/2)t} \end{aligned}$$

Plugging the initial values into this general solution gives

$$x(0) = C_1 + C_2 = 3 \qquad y(0) = 2C_1 - 2C_2 = 5.$$

These equations have solution $C_1 = \frac{11}{4}$ and $C_2 = \frac{1}{4}$. Then the solution of the initial value problem is

$$x(t) = \frac{11}{4}e^{-(1/6)t} + \frac{1}{4}e^{-(1/2)t} \ , \qquad y(t) = \frac{11}{2}e^{-(1/6)t} - \frac{1}{2}e^{-(1/2)t} \ .$$

As $t \to \infty$, the functions $x(t)$ and $y(t)$ in these solutions (asymptotically) approach 0. (The critical point (0,0) for this linear system is a node sink.)

**Exercises:**

(1) Find the general solution to the linear system if $b$ is zero. *Hint:* The first DE has no $y$'s, so solve it first and plug that solution into the second DE.

For exercises (2)-(6), find the general solution (or IVP solution when initial values are given), draw the phase plane direction field, and discuss the stability of the critical point (0,0):

(2)=(4.1.15) $\dfrac{dx}{dt} = \dfrac{1}{2}y \ , \qquad \dfrac{dy}{dt} = -8x \ .$

(3)=(4.2.3) $\dfrac{dx}{dt} = -3x + 2y \ , \qquad \dfrac{dy}{dt} = -3x + 4y \ ; \qquad x(0) = 0 \ , \qquad y(0) = 2 \ .$

(4)=(4.2.6) $\dfrac{dx}{dt} = x + 9y \ , \qquad \dfrac{dy}{dt} = -2x - 5y \ ; \qquad x(0) = 3 \ , \qquad y(0) = 2 \ .$

(5)=(6.1.18) $\dfrac{dx}{dt} = -y \ , \qquad \dfrac{dy}{dt} = 4x \ .$

(6)=(6.1.20) $\dfrac{dx}{dt} = y \ , \qquad \dfrac{dy}{dt} = -5x - 4y \ .$

Exercises (7)-(8) deal with a two-tank system. Fresh water flows into tank A, and brine flows from tank B onto the ground, both at a rate of $2r$ L/min. Brine flows from tank B to tank A at a rate of $r$ L/min, and brine flows from tank A to tank B at a rate of $3r$ L/min. The initial amounts $x(0)$ and $y(0)$ of salt in tanks A and B, respectively, are given, as are the volumes $V_A$ and $V_B$ in the tanks. Set up and solve the initial value problem modeling the situation in each exercise.

(7)=(4.2.30) $r = 10$ L/min $\quad x(0) = 0.5$ g, $\quad y(0) = 0.5$ g, $\quad V_A = 100$ L, $\quad V_B = 200$ L.

(8) $r = 3$ L/min $\quad x(0) = 1$ g, $\quad y(0) = 2$ g, $\quad V_A = 20$ L, $\quad V_B = 20$ L.

**Math 221 - Section 5**

**A quick guide to homogeneous linear systems of DEs and the phase plane**

A *standard form homogeneous* linear system of two first order differential equations is

$$\frac{dx}{dt} = ax + by \qquad \frac{dy}{dt} = cx + dy$$

where $a$, $b$, $c$, and $d$ are constants.

Homogeneous systems always have a critical point at $(0, 0)$; that is, the constant functions $x(t) \equiv 0$ and $y(t) \equiv 0$ are solutions of this system for any values of $a$, $b$, $c$, and $d$.

To solve the system when $b \neq 0$, we solve the first DE for $y$ and using this to eliminate the variable $y$ from the second DE:

$$\boxed{y = \frac{1}{b}(x' - ax)} \qquad \boxed{x'' - (a + d)x' + (ad - bc)x = 0\ .}$$

The behavior around the critical point (0,0) depends on the characteristic roots of the auxiliary equation. For example, the critical point can be stable, unstable (a source), or asymptotically stable (a sink). The critical point is described in various ways as a node, saddle, spiral, center, or star. See the pictures on the last page of this handout for pictorial definitions of these terms. **Exercise (1):** In Example (A) of the Solving linear systems handout we found that the general solution to the linear system in $\frac{dx}{dt} = x+3y$ , $\frac{dy}{dt} = 0x+2y$ is

$x(t) = 3C_1e^{2t} + C_2e^t \qquad y(t) = C_1e^{2t}$ .

Graph the two solutions where $C_1 = 0$ and $C_2 = 1$ and where $C_1 = -1$ and $C_2 = 0$ in the phase plane. What does this tell you about $(0, 0)$? **Example (B):** In Example (B) of the Solving linear systems handout we found that the general solution to the linear system $\frac{dx}{dt} = x + 2y$ , $\frac{dy}{dt} = -5x - y$ is

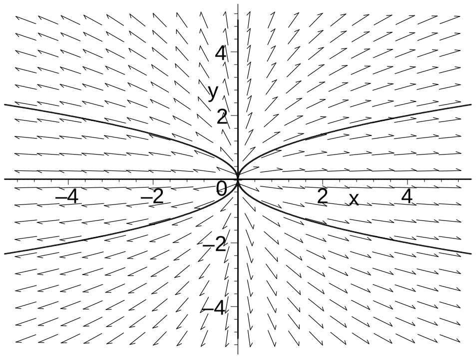
$x(t) = C_1 \sin(3t)+C_2 \cos(3t)$ , $\quad y(t) = C_1 \left(\frac{3}{2} \cos(3t) - \frac{1}{2} \sin(3t)\right)+C_2 \left(-\frac{1}{2} \cos(3t) - \frac{3}{2} \sin(3t)\right)$ .

This general solution involves only cosine and sine functions, but not exponential functions, so $x(t)$ and $y(t)$ can't (asymptotically) approach 0, and they also can't go to $\pm\infty$. The critical point (0,0) for this linear system is a stable center. **Example (C-D):** In Examples (C-D) of the Solving linear systems handout we found that the general solution to the linear system $\frac{dx}{dt} = -\frac{1}{3}x + \frac{1}{12}y$ , $\frac{dy}{dt} = \frac{1}{3}x - \frac{1}{3}y$ is

$x(t) = C_1e^{-1/6)t} + C_2e^{-(1/2)t} \qquad y(t) = 2C_1e^{-(1/6)t} - 2C_2e^{-(1/2)t}$ .

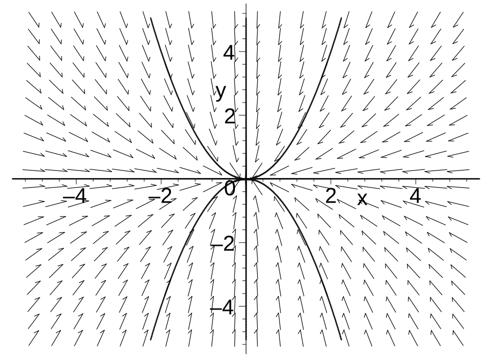
As $t \to \infty$, the functions $x(t)$ and $y(t)$ in these solutions (asymptotically) approach 0. The critical point (0,0) for this linear system is a node sink. **Exercises:** For the following systems, find the general solution and discuss the stability of the critical point $(0, 0)$: (2) $\dfrac{dx}{dt} = 4x$ , $\dfrac{dy}{dt} = x + 4y$ . (3) $\dfrac{dx}{dt} = 4x - 5y$ , $\dfrac{dy}{dt} = 2x + 6y$ .

distinct real roots, both positive



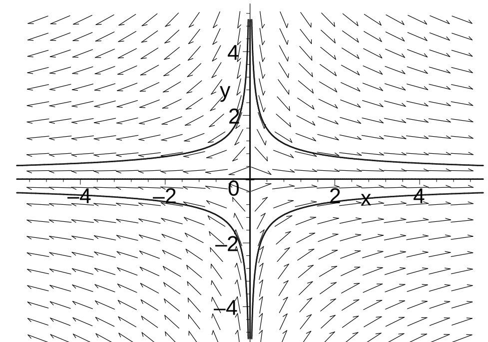

node source (unstable)

distinct real roots, both negative



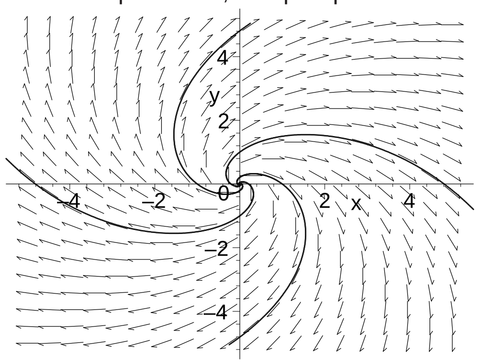

node sink (asymptotically stable)

distinct real roots, different signs



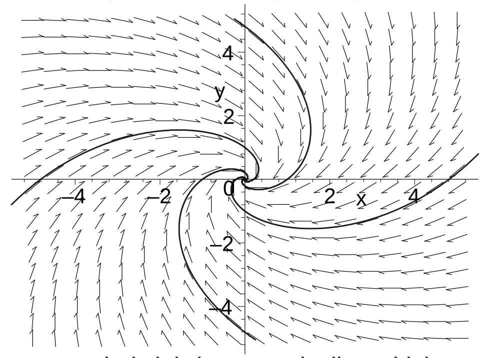

saddle point (unstable)

complex roots, real part positive



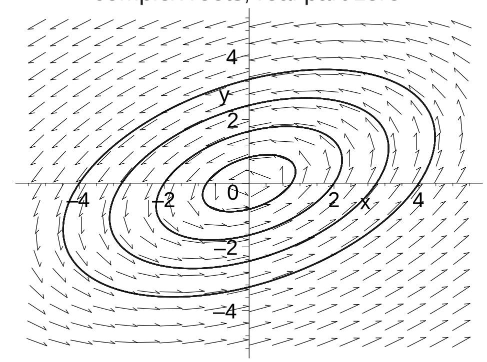

spiral source (unstable)

complex roots, real part negative



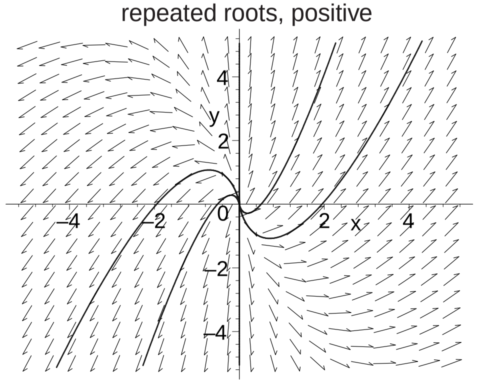

spiral sink (asymptotically stable)

complex roots, real part zero



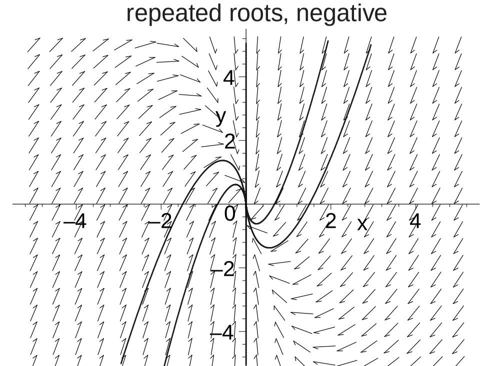

center (stable)

repeated roots, positive



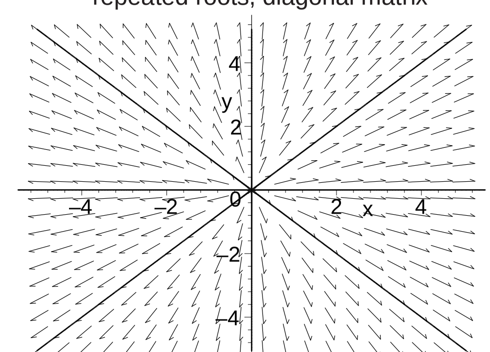

improper node, source (unstable)

repeated roots, negative




improper node, sink (asymptotically stable)

repeated roots, diagonal matrix




star point