Math 221

Topics since the second exam

Laplace Transforms.

There is a whole different set of techniques for solving n-th order linear equations, which are based on the *Laplace transform* of a function. For a function $f(t)$, it's Laplace transform is

$$\mathcal{L}\{f\} = \mathcal{L}\{f\}(s) = \int_0^\infty e^{-st} f(t)\ dt$$

The domain of $\mathcal{L}\{f\}$ is all values of s where the improper integral converges. For most basic functions f, $\mathcal{L}\{f\}$ can be computed by integrating by parts. A list of such transforms can be found on the handout from class. The most important property of the Laplace transform is that it *turns differentiation into multiplication by* s. that is:

$$\mathcal{L}\{f'\}(s) = s\mathcal{L}\{f\}(s) - f(0)$$

more generally, for the n-th derivative:

$$\mathcal{L}\{f^{(n)}\}(s) = s^n\mathcal{L}\{f\}(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0)$$

The Laplace transform is a *linear operator* in the same sense that we have used the term before: for any functions f and g, and any constants a and b,

$$\mathcal{L}\{af + bg\} = a\mathcal{L}\{f\} + b\mathcal{L}\{g\}$$

(since integration is a linear operator). We can therefore use Laplace transforms to solve linear (inhomogeneous) equations (with constant coefficients), by applying $\mathcal{L}$ to both sides of the equation:

$$ay'' + by' + cy = g(t)$$

becomes

$$(as^2 + bs + c)\mathcal{L}\{y\} - asy(0) - ay'(0) - by(0) = \mathcal{L}\{g\},\ \text{i.e.}$$

$$\mathcal{L}\{y\} = \frac{\mathcal{L}\{g\}(s) + asy(0) + ay'(0) + by(0)}{as^2 + bs + c}$$

So to solve our original equation, we need to find a function y whose Laplace transform is this function on the right. It turns out there is a formula (involving an integral) for the *inverse Laplace transform* $\mathcal{L}^{-1}$, which in principle will solve our problem, but the formula is too complicated to use in practice. Instead, we will develop techniques for recognizing functions as linear combinations of the functions appearing as the right-hand sides of the formulas in our Laplace transform tables. Then the function y we want is the corresponding combination of the functions on the left-hand sides of the formulas, because the Laplace transform is linear! Note that this approach incorporates the initial value data $y(0), y'(0)$ into the solution; it is naturally suited to solving initial value problems.

To do this we need to start with a collection of functions whose Laplace transforms we have computed; this should include the kinds of functions we have found as solutions to DEs we have so far encountered, namely products made out of the functions t^n, e^{at}, and $\sin(bt)$ or $\cos(bt)$. These have been assembled in our standard Laplace transforms table.

Along the way to building our table of transforms, we learned several more general rules:

If $\mathcal{L}\{f(t)\} = F(s)$ then $\mathcal{L}\{f'(t)\} = sF(s) - f(0)$
If $\mathcal{L}\{f(t)\} = F(s)$ then $\mathcal{L}\{e^{at}f(t)\} = F(s-a)$
If $\mathcal{L}\{f(t)\} = F(s)$ then $\mathcal{L}\{tf(t)\} = -F'(s)$

Our basic technique for using our table of Laplace transforms to find solutions is *partial fractions*: we will content ourselves with a simplified form of it, sufficient for solving second order equations. The basic idea is that we need to find the inverse Laplace transform of a function having a quadratic polynomial $as^2 + bs + c$ in its denomenator. Partial fractions tells us that, if we can factor $as^2 + bs + c = a(x - r_1)(x - r_2)$, where $r_1 \neq r_2$, then any function

$$\frac{ms+n}{as^2+bs+c} = \frac{A}{s-r_1} + \frac{B}{s-r_2}$$

for appropriate constants A and B. We can find the constants by writing

$$\frac{A}{s-r_1} + \frac{B}{s-r_2} = \frac{A(s-r_2)+B(s-r_1)}{(s-r_1)(s-r_2)} = \frac{Aa(s-r_2)+Ba(s-r_1)}{as^2+bs+c}$$

so we must have $ms + n = Aa(s - r_2) + Ba(s - r_1)$; setting the coefficients of the two linear functions equal to one another, we can solve for A and B. We can therefore find the inverse Laplace transform of $(ms + n)/(as^2 + bs + c)$ as a combination of the inverse transforms of $(s - r_1)^{-1}$ and $(s - r_2)^{-1}$, which can be found on the tables!

If $r_1 = r_2$, then we instead write

$$\frac{ms+n}{as^2+bs+c} = \frac{A}{s-r_1} + \frac{B}{(s-r_1)^2} = \frac{a(A(s-r_1)+B)}{a(s-r_1)^2} = \frac{a(A(s-r_1)+B)}{as^2+bs+c}$$

anbd solve for A and B as before.

Finally, if we *cannot* factor $as^2 + bs + c$ (i.e, it has complex roots), we can then write it as (a times) a sum of squares, by completing the square:

$as^2 + bs + c = a((s - \alpha)^2 + \beta^2)$, so

$$\frac{ms+n}{as^2+bs+c} = \frac{A\beta}{a((s-\alpha)^2+\beta^2)} + \frac{B(s-\alpha)}{a((s-\alpha)^2+\beta^2)} =$$
$$\frac{A}{a}\frac{\beta}{(s-\alpha)^2+\beta^2} + \frac{B}{a}\frac{s-\alpha}{(s-\alpha)^2+\beta^2}$$

for appropriate constants A and B (which we solve for by equating the numerators), and so it is a linear combination of $\dfrac{\beta}{(s-\alpha)^2+\beta^2}$ and $\dfrac{(s-\alpha)}{(s-\alpha)^2+\beta^2}$, both of which appear on our tables!

Handling higher degree polynomials in the denomenator is similar; if all roots are real and distinct, we write our quotient as a linear combination of the functions $(s-r_i)^{-1}$, combine into a single fraction, and set the numerators equal; if we have repeated roots, we include terms in the sum with successively higher powers $(s - r_i)^{-k}$ (where k runs from 1 to the multiplicity of the root). Complex roots are handled by inserting the term we dealt with above into the sum.

Discontinuous external force.

One area in which Laplace transforms provide a better framework for working out solutions than our "auxiliary equation" approach is when we are trying to solve an equation

$$ay'' + by' + cy = g(t)$$

where $g(t)$ is *discontinuous*, or defined in *pieces* over different time intervals. The model for a discontinuous function is the step function $u(t)$:

$$u(t) = \begin{cases} 1 & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases}$$

More generally, the function $u(t-a)$ has

$$u(t-a) = \begin{cases} 1 & \text{if } t \geq a \\ 0 & \text{if } t < a \end{cases}$$

So, for example, the function which is t for $3 \leq t \leq 5$, and is 0 everywhere else, can be expressed as $g(t) = t(u(t-3) - u(t-5))$. We can streamline things somewhat by writing $u(t-a) - u(t-b) = \chi_{[a,b]}(t) =$ the characteristic function of the interval $[a,b]$; it is 1 between a and b, and 0 everywhere else. So, for example, the piecewise-defined function

$$f(t) = \begin{cases} t & \text{if } 0 \leq t \leq 2 \\ 5-t & \text{if } 2 < t < 5 \\ 3 & \text{if } t \geq 5 \end{cases}$$

can be expressed as

$$\begin{aligned} f(t) &= t\chi_{[0,2]}(t) + (5-t)\chi_{[2,5]}(t) + 3\chi_{[5,\infty)}(t) \\ &= t(u(t) - u(t-2)) + (5-t)(u(t-2) - u(t-5)) + 3u(t-5) \end{aligned}$$

We can find the Laplace transform of such a function by finding the transform of functions of the form $f(t)u(t-a)$, which we can do directly from the integral, by making the substitution $x = t - a$:
$\mathcal{L}\{f(t)u(t-a)\} = \int_0^\infty e^{-st}f(t)u(t-a)\ dt = \int_a^\infty e^{-st}f(t)\ dt = \int_0^\infty e^{-s(t+a)}f(t+a)\ dt = e^{-as}\int_0^\infty e^{-st}f(t+a)\ dt = e^{-as}\mathcal{L}\{f(t+a)\}$.
Turning this around, we find that the inverse Laplace transform of the function $e^{-as}\mathcal{L}\{f\}(s)$ is $f(t-a)u(t-a)$. So if we can find the inverse transform of a function $F(s)$ (in our tables), this tells us how to find the inverse transform of $e^{-as}F(s)$. This is turn gives us a method for solving any initial value problem, in principle, whose inhomogeneous term $f(t)$ has finitely many values where it is discontinuous, by writing $f(t)$ as a sum of functions of the form $f_i(t)u(t-a_i)$, as above.

For example, to find the solution to the differential equation

$y'' + 2y' + 5y = g(t)$, $y(0) = 2$, $y'(0) = 1$, where $g(t)$ is the function which is 5 for $2 \leq t \leq 4$ and 0 otherwise, we would (after taking Laplace transforms and simplifying) need to find the inverse Laplace transform of the function

$$F(s) = \frac{2s+5}{s^2+2s+5} + \frac{5(e^{-2s} - e^{-4s})}{s(s^2+2s+5)}$$

Applying our partial fractions techniques, we find that

$$F(s) = 2\frac{s+1}{(s+1)^2+2^2} + \frac{3}{2}\frac{2}{(s+1)^2+2^2} + \left(\frac{1}{s} - \frac{s+1}{(s+1)^2+2^2}\frac{1}{2} - \frac{2}{(s+1)^2+2^2}\right)e^{-2s}$$
$$- \left(\frac{1}{s} - \frac{s+1}{(s+1)^2+2^2} - \frac{1}{2}\frac{2}{(s+1)^2+2^2}\right)e^{-4s}$$

We can apply $\mathcal{L}^{-1}$ to each term, using $\mathcal{L}^{-1}\{e^{-as}\mathcal{L}\{f\}(s)\} = f(t-a)u(t-a)$ for the last 6 terms (since after removing e^{-2s} and e^{-4s} the remainder of each term is in our tables). For example,

$\mathcal{L}^{-1}\{\frac{s+1}{(s+1)^2+2^2}e^{-2s}\} = e^{-(t-2)}\cos(2(t-2))u(t-2)$. The final solution, as the interested reader can work out, is

$$y = 2e^{-t}\cos(2t) + \frac{3}{2}e^{-t}\sin(2t)$$
$$+ \left[1 - e^{-(t-2)}\cos(2(t-2)) - \frac{1}{2}e^{-(t-2)}\sin(2(t-2))\right]u(t-2)$$
$$- \left[1 - e^{-(t-4)}\cos(2(t-4)) - \frac{1}{2}e^{-(t-4)}\sin(2(t-4))\right]u(t-4)$$

Isolating external force and initial conditions.

Given an n^{th} order, inhomogeneous, constant coefficients, linear initial value problem,

$$L(x) = f(t)\ ,\ x(0) = a_0, \ldots, x^{(n-1)}(0) = a_{n-1}$$

if we apply the Laplace transform to both sides, we get an equation $P(s)\mathcal{L}\{x\} - I(x) = \mathcal{L}\{f(t)\} = F(s)$, where $P(s)$ is the auxiliary polynomial for the DE, and $I(s)$ is a polynomial (of degree $n-1$) whose coefficients are determined by the the initial values $a_0, \ldots, a_{n-1}$. Solving this equation for $\mathcal{L}\{x\}$, we get

$$\mathcal{L}\{x\} = \frac{F(s)}{P(s)} + \frac{I(s)}{P(s)} = A(s) + B(s)$$

and so our solution $x(t)$ is the sum of the inverse Laplace transforms of A and B. If we have $f(t) = 0$, then $F(s) = 0$, so $A(s) = 0$, so $\mathcal{L}^{-1}\{A(s)\} = 0$; this means that $\mathcal{L}^{-1}\{B(s)\}$ is really just the solution to the associated homogeneous IVP. On the other hand, if all of the initial values are 0 (we have "trivial" initial conditions), then $I(s) = 0$, so $B(s) = 0$, so $\mathcal{L}^{-1}\{B(s)\} = 0$; this means that $\mathcal{L}^{-1}\{A(s)\}$ is the solution to our inhomogenous DE with trivial initial conditions, i.e., it is precisely the particular solution with trivial initial conditions. This has the effect of isolating the initial conditions from the inhomogeneity term (in the language of spring-mass problems, the external force term), and allows to solve our IVP by solving these two problems separately.

This leads to a viewpoint on solving IVPs using Laplace transforms known as **Duhamel's principle.** The idea is that the particular solution with trivial initial conditions is precisely

$$\mathcal{L}^{-1}\{\frac{F(s)}{P(s)}\} = \mathcal{L}^{-1}\{\frac{1}{P(s)}F(s)\}$$

But we know that $\mathcal{L}^{-1}\{F(s)\} = f(t)$, and since $P(s)$ is a polynomial, we can use partial fractions methods to compute $\mathcal{L}^{-1}\{1/P(s)\} = g(t)$. What we would like is a way to compute the inverse Laplace transform of the *product* of $F(s)$ and $1/P(s)$ in terms of

f and g. And it turns out that we can, that is, we can find a function whose Laplace transform is the product of $F(s)$ and $1/P(s)$. It is the *convolution* of f and g.

More precisely, if $\mathcal{L}\{f(t)\} = F(s)$ and $\mathcal{L}\{g(t)\} = G(s)$, then $\mathcal{L}\{(f * g)(t)\} = F(s)G(s)$, where

$$(f * g)(t) = \int_0^t f(u)g(t-u)\ du$$

The convolution product satisfies $(f * g)(t) = \int_0^t f(u)g(t-u)\ du = \int_0^t g(v)f(t-v)\ dv = (g * f)(t)$ (as the substitution $v = t - u$ will verify), so we can place the term $t - u$ in either function, as an aid to finding the quicker way to evaluate this integral in practice. Putting this together with our observations above, the solution to any linear constant coefficient IVP $L(x) = f(t)$ is

$$x(t) = (f * g)(t) + [\text{the solution to the corresponding } \textit{homogeneous} \text{ IVP}]$$

Where $g(t)$ is the inverse Laplace transform of 1/[the auxiliary polynomial of the DE]. This is Duhamel's principle. This formulation is especially useful when you have the solution to the homogenous IVP (say, using Laplce transforms!, or one of our earlier methods), and wish to determine the effects of changing the external force term $f(t)$ on the solution to the IVP (say, for designing a system to cancel out the vibrations in some piece of equipment). In a damped spring-mass system, for example, the homogenous solution will tend to 0 as $t \to \infty$, and all that will be left in the long run is $(f * g)(t)$. In general, when our other methods for computing particular solutions fail (i.e., give difficult integrals), so will this! But we can readily employ numerical methods (the trapezoid or Simpson's rule) to approximate $(f * g)(t)$, in order to understand the solutions.