Math 221

Section 5

A review of partial fractions for Laplace transforms

In solving differential equations by the method of Laplace transforms, we will repeatedly find ourselves needing to find the inverse Laplace transforms of *rational functions*, that is, quotients of polynomials. Calculus provides us with a general method of expressing such functions as sums of more basic ones, called *partial fractions.*

The basic idea is that we will work with a rational function

$$f(x) = \frac{p(x)}{q(x)} = \frac{a_0x^n + \cdots + a_{n-1}x + a_n}{b_0x^m + \cdots + b_{m-1}x + b_m}$$

by writing it as a sum of simpler functions.

The main fact that will make this (at least in theory, and often in practice) possible is that *any polynomial with real coefficients can (in principle) be expressed as the product of linear and (irreducible) quadratic polynomials.* By factoring the quotient polynomial $q(x)$ in this way, we can determine what the simpler functions will look like; a procedure very much like undetermined coefficients will then allow us to determine exactly what the simpler pieces are.

The basic procedure goes like this: starting with $f(x) = \dfrac{p(x)}{q(x)}$

(0): Make sure that degree(p)<degree(q); do polynomial long division if it isn't. I.e., write $p(x) = a(x)q(x) + b(x)$, with degree(b)<degree(q), and then

$$\frac{p(x)}{q(x)} = a(x) + \frac{b(x)}{q(x)},$$

and we can integrate $a(x)$, since it is a polynomial.

(1): Factor $q(x)$ into linear and irreducible quadratic factors. This is the only step where we really have no general procedure (basically, because there *is* none). As with the auxiliary equations for higher order DE's, we can try to find roots of the polynomial to determine linear factors, and there are procedures (like the rational roots theorem) for determining good candidates. And a good computer algebra system can give us good approximations to roots (and quadratic factors).

(2): Group common factors together as powers; if, e.g., 3 is a root of $q(x)$ four times, then we treat the four factors in what follows as giving one factor of $(x-3)^4$.

(3a): For each group $(x-a)^k$, we add together:

$$\frac{a_1}{x-a} + \cdots + \frac{a_k}{(x-a)^k}$$

These are the simpler pieces that the factor $(x-a)^k$ will contribute to the final sum.

(3b): For each group $(ax^2+bx+c)^k$, we add together:

$$\frac{c_1x+b_1}{ax^2+bx+c} + \cdots + \frac{c_kx+b_k}{(ax^2+bx+c)^k}$$

(4) Set $f(x) =$ the sum of all of these sums; solve the resulting equation for the 'undetermined' coefficients a_i, c_j , etc.

Note that we need to use *different* names for the coefficients, for each piece!

Showing *why* this procedure works (i.e., why a rational function can always be expressed as such a sum) would take us too far afield (and isn't even really about Laplace transforms!), so we will content ourselves to just use this remarkable fact. There are two basic methods for carrying out step (4), to solve for the undetermined coefficients:

In both, we put the entire sum over a common denomenator (which, it turns out, will (almost: up to multiplication by a constant) be equal to $q(x)$, if you put things over the *smallest* common denomenator) and set the resulting numerator equal to (the appropriate constant multiple of) $p(x)$.

(a) (this always works): Multiply out the numerator to a single polynomial, and set the coeffs of the two polynomials equal to one another. (This works because two polynomials are the same precisely when they have the same degree and their coefficients are equal to one another.) This gives us a system of linear equations involving the unknown coefficients, which we can solve.

Ex: $x+3 = a(x-1)+b(x-2) = (a+b)x+(-a-2b)$; solve $1 = a+b$, $3 = -a-2b$

(b) Don't multiply out the numerator! Leave it as a sum of products of terms from the denomenators. We can determine many of the unknown coefficients by plugging well-chosen values in for x.

For each linear term $(x-a)^k$, plug $x=a$ into both sides. Most of the terms of the sum will have a factor of $(x-a)$ and so will give zero, which will allow us to quickly solve for one of the coefficients.

If $k \geq 2$, take derivatives of both sides! Then by plugging in x=a, we will quickly solve for another coefficient.

Ex: $$\frac{x^2}{(x-1)^2(x^2+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+1}$$
$$= \frac{A(x-1)(x^2+1)+B(x^2+1)+(Cx+D)(x-1)^2}{(x-1)^2(x^2+1)}$$

so $A(x-1)(x^2+1)+B(x^2+1)+(Cx+D)(x-1)^2 = x^2$.

Set $x=1$, get $2B=1$, solve for B.

Take derivatives: $A(x^2+1)+A(x-1)(2x)+B(2x)+C(x-1)^2+2(Cx+D)(x-1) = 2x$
Set $x=1$, get $2A+0+2B+0+0=2$, solve for A (since we already know B)

The end result of this process is a an expression for $\frac{p(x)}{q(x)}$ as a sum of rational functions of the form

$$\frac{a_i}{(x-a)^i} \text{ and } \frac{c_ix+b_i}{(x^2+bx+c)^i} = \frac{c_ix+b_i}{((x+\alpha)^2+\beta^2)^i}$$

These can then be dealt with one at a time; finding the inverse Laplace transform of pieces of the second type can be simplified by writing

$$\frac{c_ix+b_i}{((x+\alpha)^2+\beta^2)^i} = c_i\frac{x+\alpha}{((x+\alpha)^2+\beta^2)^i} + \frac{b_i-c_i\alpha}{\beta}\frac{\beta}{((x+\alpha)^2+\beta^2)^i}$$

At least for $i=1$ ($i>1$ requires further techniques, e.g., convolution), the first term has inverse Laplace transform a multiple of $e^{\alpha t}\cos(\beta t)$; the second term has inverse Laplace transform a multiple of $e^{\alpha t}\sin(\beta t)$.