Math 208H, Section 1

Some practice problems for the Final Exam

A1. Find the length of the parametrized curve

$$\vec{r}(t) = (t^6 \cos t, t^6 \sin t) \quad , \quad 0 \le t \le \pi$$

A2. Find the equation of the plane tangent to the graph of

$$z = f(x,y) = xe^y - \cos(2x+y)$$

at $(0,0,-1)$

In what direction is this plane tilting up the most?

A3. Find the critical points of the function

$$z = g(x,y) = x^2y^3 - 3y - 2x$$

and for each, determine if it is a local max, local min, or saddle point.

A4. Find the integral of the function

$$z = h(x,y) = \ln(x^2+y^2+1)$$

over the region

$$R = \{(x,y) : x^2+y^2 \le 4\}$$

A5. Find the integral of the function

$$k(x,y,z) = z$$

over the region lying inside of the sphere of radius 2 (centered at the origin $(0,0,0)$) and above the plane $z = 1$.

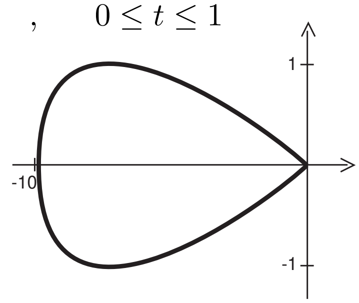
A6. Show that the vector field $\vec{F} = \langle y^2, 2xy - 1 \rangle$ is conservative, find a potential function $z = f(x,y)$ for $\vec{F}$, and use this potential function to (quickly!) find the integral of $\vec{F}$ along the path

$$\vec{r}(t) = (t\sin(2\pi t) - e^t, \ln(t^2+1) - 5t^2) \quad , \quad 0 \le t \le 1$$

A7. Use Green's Theorem to find the area of the region enclosed by the curve

$\vec{r}(t) = (t^2 - 2\pi t, \sin t)$

$0 \le t \le 2\pi$



A8. Find the flux of the vector field

$\vec{G} = \langle x^2, xz, y \rangle$ through that part of the graph of

$$z = f(x,y) = xy$$

lying over the rectangle

$$1 \le x \le 3 \quad , \quad 0 \le y \le 3$$

B1. Find the orthogonal projection of the vector $\vec{v}$ = (3,1,2) onto the vector $\vec{w}$ = (-1,4,2) .

B2. Find the equation of the plane passing through the points

(1,1,1), (2,1,3), and (-1,2,1)

B4. Find the integral of the function $f(x,y) = xy^2$
over the region in the plane lying between the graphs of
$a(x) = 2x$ and $b(x) = 3 - x^2$

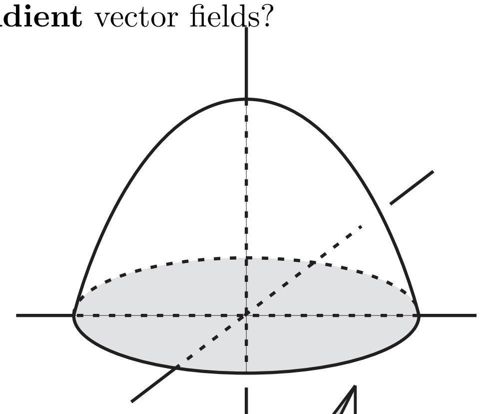
B5. Find the integral of the vector field $F(x,y) = (xy, x+y)$
along the parametrized curve $\vec{r}(t) = (e^t, e^{2t})$ $0 \le t \le 1)$.

B6. Which of the following vector fields are **gradient** vector fields?

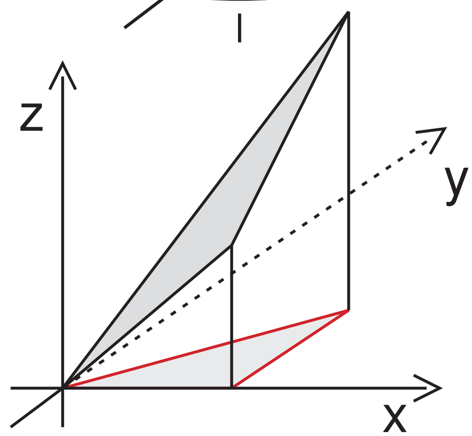
(a) $F(x,y) = (y\sin(xy), x\sin(xy))$

(b) $G(x,y,z) = (x^2y, z^2 + x, 2yz)$

(c) $H(x,y,z) = (y + y^2z, x + 2xyz, xy^2)$



B7. Use the Divergence Theorem to find the flux integral of the vector field
$F(x,y,z) = (y, xy, z)$ through the boundary of the region lying under the graph of
$f(x,y) = 1 - x^2 - y^2$ and above the x-y plane (see figure).



B8. Use Stokes Theorem to find the line integral of the vector field
$F(x,y,z) = (xy, xz, yz)$
around the triangle with vertices
(0,0,0), (1,0,1), and (1,1,2)
(see figure).

B9. Imagine a box with side lengths $x = 2$, $y = 3$, and $z = 4$, and these lengths all change with time. How fast is the volume of the box changing, if

$$\frac{dx}{dt} = 3,\ \frac{dy}{dt} = -2, \text{ and } \frac{dz}{dt} = -1\ ?$$

B10. Find the critical points of the function

$$f(x,y) = x^3y^2 - 6x^2 - y^2$$

and for each, determine if it is a rel max, rel min, or saddle point. Does the function have a global maximum?

B11. By switching the order of integration, find the integral

$$\int_0^1 \int_x^1 xe^{\frac{x^2}{y}}\ dy\ dx$$

B13. Find the flux integral of the vector field

$$F(x, y, z) = (1, y^2, xz)$$

over the sphere of radius 1 centered at (0,0,0) .

C1. Find the equation of the plane tangent to the graph of the function

$f(x, y) = \sqrt{2x^2 + y} = (2x^2 + y)^{\frac{1}{2}}$ at the point $(2, 1, 3)$.

C2. If the temperature in a room is given by the function $H(x, y, z) = \dfrac{xy + z}{x + y}$,

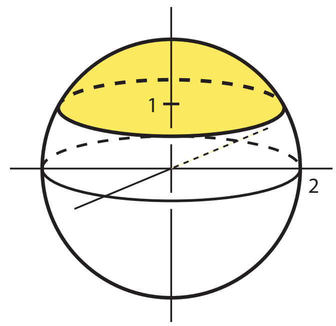
use the Chain Rule to compute the *rate of change* of the temperature, as you travel along the curve $\gamma(t) = (x(t), y(t), z(t)) = (t^2, 2t, t^3)$, at time $t = 1$.

C3. Find the point(s) on the ellipse $3x^2 + y^2 = 1$ where the function $f(x, y) = x^3 y$ has its smallest (i.e., most negative) value.

C4. By *reversing* the order of integration, compute

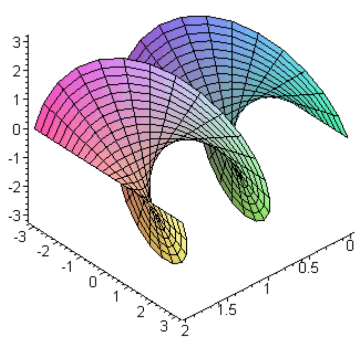
$$\int_0^2 \int_x^2 x\sqrt{y^3 + 1}\, dy\, dx$$

C5. Set up **but do not compute** the triple integrals needed to find the volume of the region lying inside of the sphere $x^2 + y^2 + z^2 = 4$ and *above* the plane $z = 1$ **in both rectangular and spherical coordinates** (see figure).

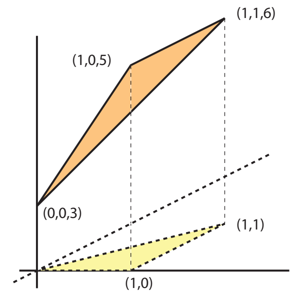


C6. Find a potential function for the conservative vector field (in the plane) $\vec{F}(x, y) = (\cos x \cos y, -\sin x \sin y)$ and use this to compute the line integral $\int_\gamma \vec{F} \circ d\vec{r}$ for the curve $\gamma(t) = (t \sin(\pi t), t^2 \cos(\pi t))$, $0 \le t \le 2$.

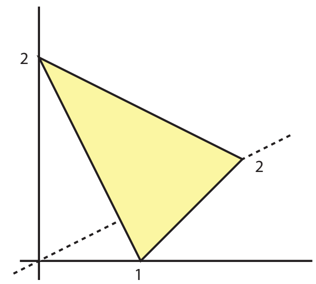
C7. Set up **but do not evaluate** an iterated integral which will compute the flux integral of the vector field $\vec{F}(x, y, z) = (y, x, z)$ across the "helical spiral" Σ, parametrized by $T(u, v) = (u, v \cos u, v \sin u)$, for $0 \le u \le 2\pi$ and $-1 \le$ (see figure).



C8. Use Stokes' Theorem to compute the work done by the force field $\vec{G}(x,y,z) = (xy, z, xz)$ around the edges of the triangle lying on the graph of the function $z = 2x + y + 3$, with corners at $(0,0,3)$, $(1,0,5)$, and $(1,1,6)$ (see figure).



C9. Use the Divergence Theorem to compute the flux of the vector field
$\vec{F}(x,y,z) = (yz, x, xz)$
through (<u>all</u> of) the sides of the "pyramid" obtained by slicing a corner off of the first octant $(x \geq 0\ ,\ y \geq 0\ ,\ z \geq 0)$ by the plane $2x + y + z = 2$ (see figure).



D1. Find the equation of the plane tangent to the graph of the function

$$f(x,y) = \frac{xy}{x+2y}$$

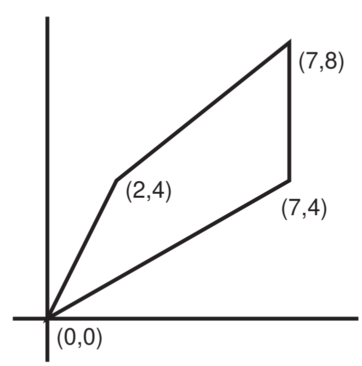
at the point $(1, 2, f(1,2))$. What vector is perpendicular to this plane?

D2. Find the directional derivative of the function $f(x,y) = xy^2 + x^2y$ in the direction of the velocity vector of the parametrized curve $\gamma(t) = (t\sin(t), 2-t)$, at time $t = \pi/2$.

D3. Recall that the line $y = L(x) = ax + b$ that 'best fits' a collection (x_i, y_i) of points is the one which minimizes the quantity $\sum_{i=1}^{n}(L(x_i) - y_i)^2$. Find the best fitting line for the points

$$(0,0)\ ,\ (1,2)\ ,\ \text{and}\ (3,2)\ .$$

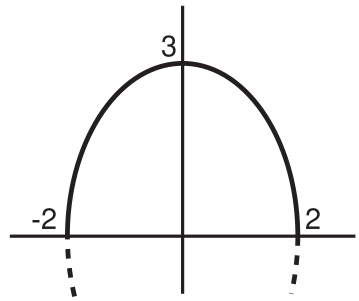
D4. Show how to express a double integral of some function $z = f(x,y)$ over the region R lying inside of the polygon shown below, as a sum of one or more iterated integrals.



D5. Find the work done by the vector field

$$\vec{F}(x,y) = (1, x^2)$$

along the top half of the ellipse given by $\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$, from $(2,0)$ to $(-2,0)$ (see figure).



D6. Show that the vector field $\quad \vec{F}(x,y) = (y + \frac{1}{x}, x + \frac{1}{y}) \quad$ is a conservative vector field, and find a potential function for $\vec{F}$.

D7. Find the flux of the vector field

$$\vec{F}(x,y,z) = (y, y, yz)$$

through the graph of the function $\quad z = f(x,y) = xy \quad$ which lies above the rectangular region R in the plane lying between the x- and y-axes and the lines $x = 3$, $y = 2$.

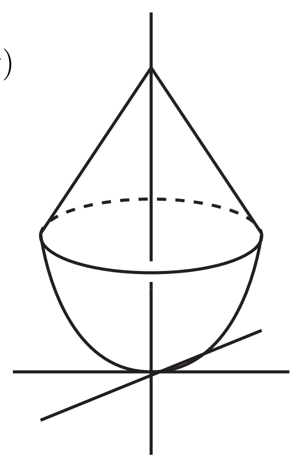
D8. Use the Divergence Theorem to **set up but not evaluate** the integral required to find the flux of the vector field

$$\text{ctln}\vec{F}(x,y,z) = (x, 2, xz)$$

through the boundary of the region lying between the graphs of the functions

$f(x,y) = x^2 + y^2 \qquad$ and $\qquad g(x,y) = 6 - \sqrt{x^2+y^2}$
(see figure!).

[Hint: to find out where the graphs meet, set $r = \sqrt{x^2+y^2}$ and solve for r...]



D9. Use the fact that
$\vec{F}(x,y,z) = (1, xy, 1 - xz) = \text{curl}(xyz, x, y)$
to use Stokes' Theorem to compute the flux integral of $\vec{F}$ over the top half of the sphere of radius 2 centered at the origin,
$\{(x,y,z) \; : \; x^2 + y^2 + z^2 = 4, z \geq 0\}$
(see figure).