Solutions

Name:

Math 208H, Section 1

Exam 2

Show all work. How you get your answer is just as important, if not more important, than the answer itself. If you think it, write it!

1. (20 pts.) Find the critical points of the function

$$f(x,y) = 2xy^2 - x^2 - 8y^2 ,$$

and for each, determine if the point is a rel max, rel min, or saddle point.

$f_x = 2y^2 - 2x = 0 \longrightarrow 2x = 2y^2 \longrightarrow x = y^2$

$f_y = 4xy - 16y = 0 \longrightarrow 4y(x-4) = 0 \longrightarrow y = 0$ or $x = 4$

$\downarrow$ $x = 0^2 = 0$; $\downarrow$ $4 = y^2 \rightarrow y = \pm 2$

[or: $4xy - 16y = 4y^3 - 16y = 4y(y^2-4) = 4y(y-2)(y+2) = 0$

$\longrightarrow y = 0, y = 2,$ or $y = -2$]

$\downarrow$ $x=0$ $\quad$ $x=4$ $\quad$ $x = (-2)^2 = 4$

Other way,

critical points are: $(0,0)$, $(\cancel{3,4}\ 4,2)$, $(4,-2)$

$f_{xx} = -2$, $f_{xy} = 4y = f_{yx}$, $f_{yy} = 4x - 16$

At $(0,0)$: $D = f_{xx}f_{yy} - (f_{xy})^2 = (-2)(-16) - (0)^2 = 32 > 0$

$f_{xx} = -2 < 0$ $\Longrightarrow$ rel max.

At $(4,2)$: $D = (-2)(0) - (8)^2 = -64 < 0$

$\Longrightarrow$ saddle point

At $(4,-2)$: $D = (-2)(0) - (-8)^2 = -64 < 0$

$\Longrightarrow$ saddle point

2. (15 pts.) Find the point(s) on the graph of the equation

$$g(x,y) = x^2 + 4y^2 = 8$$

that **minimizes** the function $f(x,y) = x + 2y$.

$\boxed{x^2+4y^2=8}$

$\nabla f = (1,2) \qquad \nabla g = (2x, 8y)$

$\nabla f = \lambda \nabla g$!

$1 = \lambda(2x) = 2\lambda x$

$2 = \lambda(8y) = 8\lambda y$

or (either way works ...)

$x = \frac{1}{2\lambda}, \ y = \frac{2}{8\lambda} = \frac{1}{4\lambda}$

$\rightarrow 8 = x^2 + 4y^2 = \frac{1}{4\lambda^2} + 4\frac{1}{16\lambda^2} = \frac{1}{2\lambda^2}$

$\rightarrow 2\lambda^2 = \frac{1}{8} \rightarrow \lambda^2 = \frac{1}{16} \rightarrow \lambda = \pm\frac{1}{4}$

$\lambda = \frac{1}{4} \rightarrow x = \frac{1}{2\cdot\frac{1}{4}} = 2, \ y = \frac{1}{4\cdot\frac{1}{4}} = 1$

$\lambda = -\frac{1}{4} \rightarrow x = \frac{1}{2\cdot\frac{-1}{4}} = -2, \ y = \frac{1}{4\cdot\frac{-1}{4}} = -1$

$\lambda = \frac{1}{2x} = \frac{2}{8y} = \frac{1}{4y}$

$\rightarrow 4y = 2x$

$\rightarrow x = 2y$

$\rightarrow (2y)^2 + 4y^2 = 8 = 8y^2$

$\rightarrow y^2 = 1 \rightarrow y = \pm 1$

$y = 1 \rightarrow x = 2y = 2$

$y = -1 \rightarrow x = 2y = -2$

$\hookrightarrow (2,1)$ or $(-2,-1)$

are where max/min occur.

$f(2,1) = 2 + 2\cdot 1 = 4$

$f(-2,-1) = -2 + 2(-1) = -4$

min!

So minimum occurs at $(-2,-1)$.

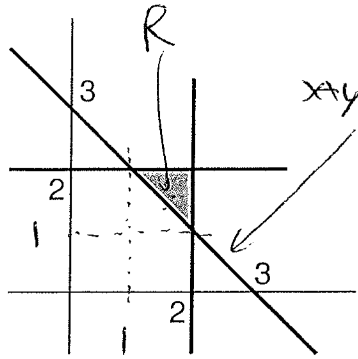
3. (15 pts.) Find the integral of the function

$$f(x,y)=\frac{x}{y}$$

over the region in the plane lying between the lines $x+y=3$, $x=2$, and $y=2$.

[Note: one iterated integral is probably less trouble than the other; which variable would you prefer to integrate <u>first</u>?]

$R:\ 3-x\le y\le 2$ for $1\le x\le 2$ or $3-y\le x\le 2$ for $1\le y\le 2$



$y=3-x$

$x+y=3$

$x=3-y$

$$\int_R f\,dA=\int_1^2\int_{3-x}^2 \frac{x}{y}\,dy\,dx$$

$$=\int_1^2 x\ln|y|\Big|_{3-x}^2\,dx \qquad dv=x\,dx,\ u=\ln(3-x)$$

$$=\int_1^2 x(\ln 2)-x\ln(3-x)\,dx$$

$$=\frac{x^2}{2}\ln 2\Big|_1^2-\left(\frac{x^2}{2}\ln(3-x)\Big|_1^2-\int_1^2\frac{-x^2}{2(3-x)}dx\right)$$

$$=(2-\tfrac{1}{2})\ln 2-\left(2\ln(1)-\tfrac{1}{2}\ln(2)-\frac{1}{2}\int_1^2\frac{x^2}{3-x}dx,\right)$$

which is getting too ugly to continue with....!

OR!

$$\int_R f\,dA=\int_1^2\int_{3-y}^2 \frac{x}{y}\,dx\,dy$$

$$=\int_1^2 \frac{x^2}{2y}\Big|_{3-y}^2\,dy$$

$$=\int_1^2 \frac{4}{2y}-\frac{(3-y)^2}{2y}\,dy$$

$$=\int_1^2 \frac{4-9+6y-y^2}{2y}\,dy$$

$$=\int_1^2 -\tfrac{1}{2}y+3-\tfrac{5}{2}\tfrac{1}{y}\,dy$$

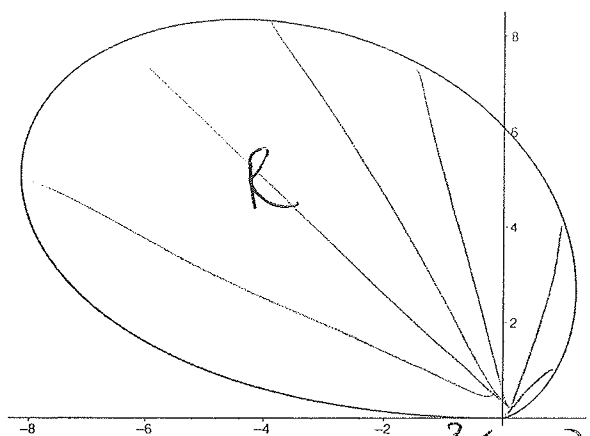
$$=-\tfrac{1}{4}y^2+3y-\tfrac{5}{2}\ln|y|\Big|_1^2$$

$$=\boxed{\left(-1+6-\tfrac{5}{2}\ln(2)\right)-\left(-\tfrac{1}{4}+3-\tfrac{5}{2}(0)\right)}$$

$$=5-\tfrac{11}{4}-\tfrac{5}{2}\ln(2)=\boxed{\tfrac{9}{4}-\tfrac{5}{2}\ln(2)}$$

4. (15 pts.) Recall that the area of a region R in the plane can be computed as the integral of the function $f(x,y)=1$ over the region. Use this, and polar coordinates, to find the area of the region lying inside of the *polar* curve

$$r=\theta^2(\pi-\theta)\ ,\ 0\le\theta\le\pi \qquad \text{(see figure)}$$



$R:\ 0\le r\le \theta^2(\pi-\theta)$

for $0\le\theta\le\pi$

$$\text{Area}=\int_R 1\,dA=\int_0^{\pi}\int_0^{\theta^2(\pi-\theta)} 1\ r\,dr\,d\theta$$

$$=\int_0^{\pi}\frac{r^2}{2}\Big|_0^{\theta^2(\pi-\theta)}d\theta=\frac{1}{2}\int_0^{\pi}\theta^4(\pi-\theta)^2\,d\theta$$

$$=\frac{1}{2}\int_0^{\pi}\theta^6-2\pi\theta^5+\pi^2\theta^4\,d\theta$$

$$=\frac{1}{2}\left(\frac{\theta^7}{7}-\frac{2\pi}{6}\theta^6+\frac{\pi^2}{5}\theta^5\right)\Big|_0^{\pi}$$

$$\boxed{=\frac{1}{2}\left(\pi^7\left(\frac{1}{7}-\frac{1}{3}+\frac{1}{5}\right)-(0-0+0)\right)}=\frac{\pi^7}{2}\left(\frac{12}{35}-\frac{1}{3}\right)$$

$$=\frac{\pi^7}{2}\left(\frac{36-35}{105}\right)=\frac{\pi^7}{2}\left(\frac{1}{105}\right)=\boxed{\frac{\pi^7}{210}}$$

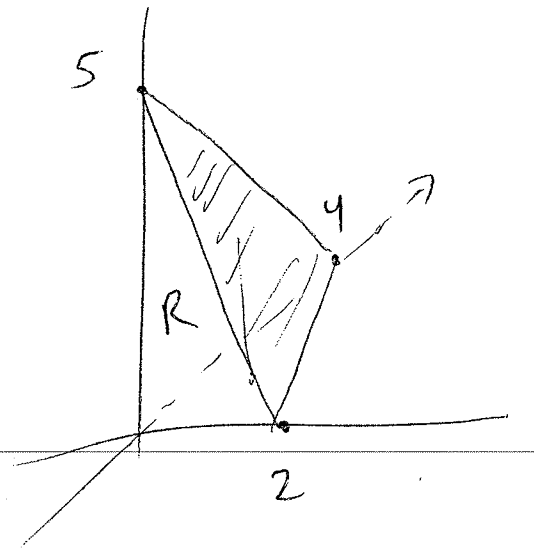
5. (15 pts.) Sketch the region involved, and set up, *but do not evaluate*, an iterated integral which will compute the integral of the function

$$f(x,y,z) = xy + z$$

over the region in 3-space lying in the first octant ($x \geq 0$, $y \geq 0$, $z \geq 0$) and below the plane

Plane! $$\frac{x}{2} + \frac{y}{4} + \frac{z}{5} = 1 \longrightarrow \text{intercepts!}$$

$(2,0,0)$
$(0,4,0)$
$(0,0,5)$



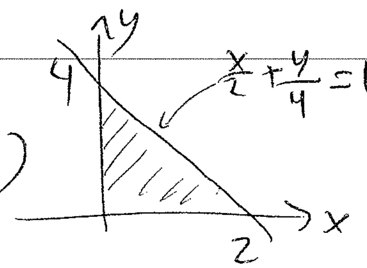
We can reasonably integrate in any order---

$$0 \leq z \leq 5\left(1 - \frac{x}{2} - \frac{y}{4}\right)$$

for shadow:

$0 \leq y \leq 4\left(1 - \frac{x}{2}\right)$

for $0 \leq x \leq 2$



$$\int_R f\,dV = \int_0^2 \int_0^{4(1-\frac{x}{2})} \int_0^{5(1-\frac{x}{2}-\frac{y}{4})} xy + z \; dz\,dy\,dx$$

or: $$\int_R f\,dV = \int_0^5 \int_0^{2(1-\frac{z}{5})} \int_0^{4(1-\frac{x}{2}-\frac{z}{5})} xy + z \; dy\,dx\,dz$$

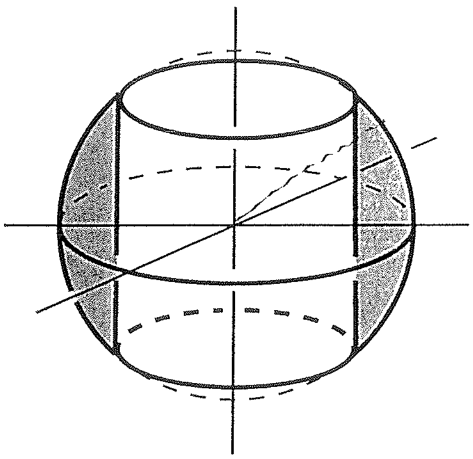
or ....

6. (20 pts.) Set up, *but do not evaluate*, the integrals, in **both** cylindrical and spherical coordinates, which will compute the integral of the function

$$f(x,y,z) = x + 3y$$

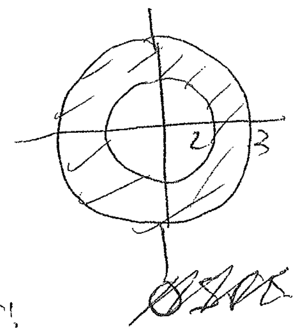
over the region in 3-space lying *inside* of the sphere $x^2+y^2+z^2=9$ of radius 3, and *outside* of the cylinder $x^2+y^2=4$ of radius 2; see figure.

[Note: at least one of your answers will involve the arcsin of a number we do not know the arcsin of....]



cylindrical! $z = \pm\sqrt{9-(x^2+y^2)} = \pm\sqrt{9-r^2}$

shadow:



In polar! $2 \le r \le 3$, $0 \le \theta \le 2\pi$

$x + 3y = r\cos\theta + 3r\sin\theta$

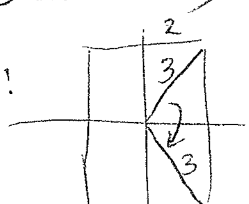
$$\boxed{\int_R f\,dV = \int_0^{2\pi}\int_2^3\int_{-\sqrt{9-r^2}}^{\sqrt{9-r^2}} (r\cos\theta + 3r\sin\theta)\, r\,dz\,dr\,d\theta.}$$

spherical! ρ goes from cylinder to sphere, $x^2+y^2=4$ to $\rho=3$.

$4 = (\rho\cos\theta\sin\varphi)^2 + (\rho\sin\theta\sin\varphi)^2 = \rho^2\sin^2\varphi \to \rho = \frac{2}{\sin\varphi}$

$\boxed{0 \le \theta \le 2\pi}$

Range of φ!



$\arcsin(\frac{2}{3})$ to $\pi - \arcsin(\frac{2}{3})$

$x + 3y = \rho\cos\theta\sin\varphi + 3\rho\sin\theta\sin\varphi$

$$\boxed{\int_R f\,dV = \int_0^{2\pi}\int_{\arcsin(\frac{2}{3})}^{\pi-\arcsin(\frac{2}{3})}\int_{2/\sin\varphi}^{3} (\rho\cos\theta\sin\varphi + 3\rho\sin\theta\sin\varphi)(\rho^2\sin\varphi)\,d\rho\,d\varphi\,d\theta}$$