Math 208H

Topics for first exam

Functions of Several Variables

Function of one variable: one number in, one out. Picture a black box; one input and one output.

Function of several variables: several inputs, one output. Picture a quantity which depends on several different quantities. E.g., distance from the origin in the plane:

$$\text{distance} = \text{d} = \sqrt{x^2 + y^2}$$

depends on both the x- and y-coordinates of our point.

Our goal is to understand functions of several variables, in much the same way that the tools of calculus allow us to understand functions of one variable. And our basic tool is going to be to *think of a function of several variables as a function of* **one variable** *(at a time!)*, so that we can use those tools to good effect.

3-space

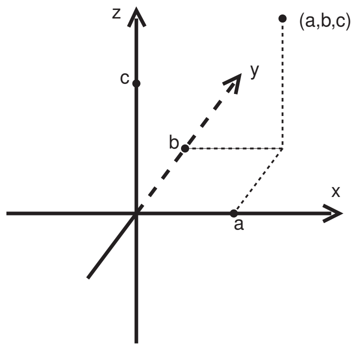
One of the best ways to understand a function is to graph it. For one variable, $y = f(x)$, this means to plot all of the pairs of points $(x, f(x))$, where x is in the domain of f. For functions of two variables, $z{=}f(x, y)$, we need to plot all *triples* $(x, y, f(x, y))$. We need to build our graphs in **3-space**.

Cartesian coordinates: three axes, each perpendicular to one another, all meeting at the *origin* (0,0,0). Axes are labelled by the *right hand rule*; the thumb, forefinger, and middle finger of the right hand point in the x, y, and z directions, respectively. The point (a, b, c) is the one reached from the origin by moving a units in the direction of the (positive) x-axis, then b units in the y direction, and then c units in the z direction.

The distance between points (a,b,c) and (x,y,z) is given by a formula very similar to the one for the plane:

$$d = \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}$$

So, e.g., the points satisfying $x^2 + y^2 + z^2 = 9$ are all of the points with distance 3 from the origin (0,0,0), i.e., they form a sphere of radius 3, centered at the origin.



Graphs of functions of two variables

How do we see what a graph looks like? Think about the function one variable at a time!

If we set $y = c$=constant, and look at $z = f(x, c)$, we are looking at a function of one variable, x, which we can (in theory) graph. This graph is what we would see when the *plane* $y = c$ meets the graph $z = f(x, y)$; this is a (vertical) *cross section* of our graph (parallel to the plane $y = 0$, the xz-plane). Similarly, if we set $x = d$=constant, and look at the graph of $z = f(d, y)$ (as a function of y), we are seeing vertical cross sections of our original graph, parallel to the yz-plane. Several of these x- and y-cross sections together can give a very good picture of the general shape of the graph of our function $z{=}f(x, y)$.

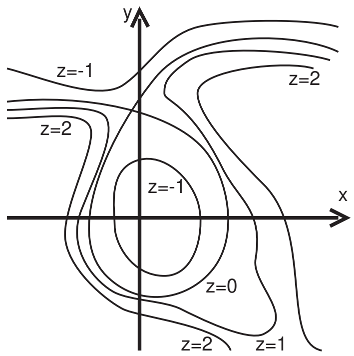
Some of the simplest functions to describe are linear functions; functions havng equations of the form $z{=}ax + by + c$. Their cross sections are all lines; the cross sections x =const all have the same slope b, and the y-cross sections all have slope a.

Another simple type of function is *cylinders*; these are functions like $f(x, y) = y^2$ which, although we think of them as functions of x and y, the output does not depend on one of the inputs. Cross sections of such functions, setting equal to a constant whichever variable is absent from the expression, will all be identical, so the graph looks like copies of the exact same function, stacked side-by-side.

Contour diagrams

The cross sections of the graph of a function are obtained by slicing our graph with vertical planes (parallel to one of our coordinate planes). But we could also use *horizontal* planes as well, that is, the planes z=constant. In other words, we graph $f(x,y) = c$ for different values of the constant c. These are called *contour lines* or *level curves* for the function f, since they represent all of the points on the graph of f which lie on the same horizontal level (the term contour line is borrowed from topographic maps; the lines represent the level curves of the height of land). These have the advantage that they can be graphed *together* in the xy-plane, for different values of c, because the level curves corresponding to different values of c cannot meet (a point (x, y) on both level curves would satisfy $c = f(x, y) = d$, so f would not be a function....)

A collection of level curves also gives a good picture of what the graph of our function f looks like; we can imagine wandering through the doman of our function, reading off the value of the function f by looking at what level curve we are standing on. We usually draw level curves for equally spaced values of c; that way, if the level curves are close together, we know that the function is changing values rapidly in that region, while if they are far appart, the values of the function are not varying by a large amount in that area.



We usually, for convenience, draw the level curves of f on a single xy-plane (since we can keep them somewhat separate), labelling each curve with its z-value. We could reconstruct a picture of the graph of f by simply drawing the level curve $f(x, y) = c$ on the horizontal plane $z = c$ in 3-space.

Linear functions

Just as lines play an important role in the 1-variable theory of calculus (e.g., as tangent lines to functions), linear function play an important role in the several variable theory.

The most general equation for a plane in 3-space is

$$ax + by + cz = d \text{ , where } a, b, c \text{ and } d \text{ are constants,}$$

although typically, our planes will come in the form $z = ax + by + c$. The number a is called the x-*slope*, since it tells us how much z changes if we move 1 unit to the right in the x-direction. For similar reasons, b is called the y-*slope*.

Typically, there is exactly one plane passing through any three particular points in 3-space (unless they happen to line up in a line, then there are many planes possible). Later we will see how to determine an equation for this plane.

We can also completely describe a plane by knowing a point (x_0, y_0, z_0) on the plane, and its x-slope m and y-slope n; the equation for the plane is then

$$z = z_0 + m(x - x_0) + n(y - y_0)$$

This will often be our method for finding equations of planes, since it is these three pieces of information which we will know when trying to compute the *tangent plane* to the graph of a function.

If we look at the level curves of a linear function, $z = ax + by + c = d =$ constant, they are a collection of parallel lines; if we choose equally spaced horizontal levels, they will be *equally spaced parallel lines.*

Functions of more than two variables

And there is no reason to stop with two variables for a function! An expression like $F = F(M, m, r) = GMm/r^2$ can be thought of as a function describing F as a function of M, m, and r (and G!). When we think of the graph of this function (as a function of the first three variables), its graph will live in 4-space! However, we can still get an impression of what the function looks like, by graphing

$F(M,m,r) = c =$ constant, for various values of c. These are *level surfaces* for the function f. We can get a picture of what the level surfaces look like by taking cross sections! Or we could look at each level surface's level *curves.*

Vectors

In one-variable calculus, we make a distinction between speed and velocity; velocity has a direction (left or right), while speed doesn't. Speed is the *size* of the velocity. This distinction is even more important in several variable calculus, and motivates the notion of a *vector.*

A vector $\vec{v}$ is an arrow pointing *from* one point in the plane (or 3-space or ...) *to* another. A vector is thought of as pointing frm its *tail* to its *head.* If it points from P to Q, we call the vector $\vec{v} = \overrightarrow{PQ}$.

A vector has both a *size* (= length = distance from P to Q) and a *direction.* Vectors that have the same size and point in the same direction are often thought of as the same, even if they have different tails (and heads). Put differently, by picking up the vector and translating it so that its tail is at the origin (0,0), we can identify $\vec{v}$ with a point in the plane, namely its head (x,y), and write $\vec{v} = (x,y)$. If $\vec{v}$ goes from (a,b) to (c,d), then we would have $\vec{v} = (c-a, d-b)$. The length of $\vec{v} = (a,b)$ is then $||\vec{v}|| = \sqrt{a^2+b^2}$.

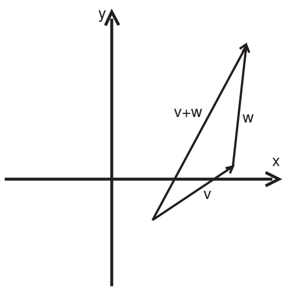
In 3-space we have three special vectors, pointing in the direction of each coordinate axis (in the plane there are, analogously, two); these are called $\vec{i}$ = (1,0,0), $\vec{j}$ = (0,1,0), and $\vec{k}$ = (0,0,1). These come in especially handy when we start to add vectors. There are several different points of view to vector addition:

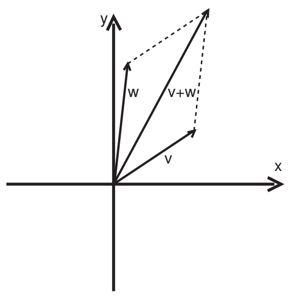
(1) move the vector $\vec{w}$ so that its head is on the tail of $\vec{v}$; then the vector $\vec{v}+\vec{w}$ has tail equal to the tail of $\vec{v}$ and head equal to the head of $\vec{w}$;

(2) move $\vec{v}$ and $\vec{w}$ so that their tails are both at the origin, and build the parallelogram which has sides equal to $\vec{v}$ and $\vec{w}$; then $\vec{v}+\vec{w}$ is the vector that goes from the origin to the opposite corner of the parallelogram;

(3) if $\vec{v} = (a,b)$ and $\vec{w} = (c,d)$, then $\vec{v}+\vec{w} = (a+c, b+d)$

We can also subtract vectors; if they share the same tail, $\vec{v}-\vec{w}$ is the vector that points from the head of $\vec{w}$ to the head of $\vec{v}$ (so that $\vec{w}+(\vec{v}-\vec{w}) = \vec{v}$). In coordinates, we simply subtract the coordinates. We can also *rescale* vectors = multiply them by a constant factor; $a\vec{v}$ = vector pointing in the same direction, but a times as long. (We use the convention that if $a<0$, then $a\vec{v}$ points in the *opposite* direction from $\vec{v}$.)





Using coordinates, this means that $a(x,y) = (ax, ay)$. To distinguish a from the coordinates or the vector, we call a a *scalar.*

All of these operations satisfy all of the usual properties you would expect:

$$\vec{v}+\vec{w} = \vec{w}+\vec{v} \qquad (\vec{v}+\vec{w})+\vec{u} = \vec{v}+(\vec{w}+\vec{u})$$
$$a(b\vec{v}) = (ab)\vec{v} \qquad a(\vec{v}+\vec{w}) = a\vec{v}+a\vec{w}$$

There is also nothing special in all of this about vectors in the plane; all of these ideas work for vectors in 3-space, or 4-space, or .....

Dot products

One thing we haven't done yet is multiply vectors together. It turns out that there are two ways to reasonably do this, serving two very different sorts of purposes.

The first, the dot product, is intended top measure the extent to which two vectors $\vec{v}$ and $\vec{w}$ are pointing in the same direction. It takes a pair of vectors $\vec{v} = (v_1, \ldots, v_n)$ and $\vec{w} = (w_1, \ldots, w_n)$, and gives us a *scalar* $\vec{v} \bullet \vec{w} = v_1 w_1 + \cdots + v_n w_n$.

Note that $\vec{v} \bullet \vec{v} = v_1^2 + \cdots + v_n^2 = ||\vec{v}||^2$. In general, $\vec{v} \bullet \vec{w} = ||\vec{v}|| \cdot ||\vec{w}|| \cdot \cos(\theta)$, where θ is the angle between the vectors $\vec{v}$ and $\vec{w}$ (when they have the same tail); this can be seen by an application of the Law of Cosines. This in turn allows us to compute this angle:

The *angle* Θ between v and w = the angle (between 0 and π with $\cos(\Theta) = \langle v, w \rangle / (||v|| \cdot ||w||)$

The dot product satisfy some properties which justify calling it a product:

$$\vec{v} \bullet \vec{w} = \vec{w} \bullet \vec{v} \quad , \quad (k\vec{v}) \bullet \vec{w} = k(\vec{v} \bullet \vec{w}) \quad , \quad \vec{v} \bullet (\vec{w} + \vec{u}) = \vec{v} \bullet \vec{w} + \vec{v} \bullet \vec{u}$$

Two vectors are orthogonal (= perpendicular) if the angle θ between them is $\pi/2$, so $\cos(\theta)$=0; this means that $\vec{v} \bullet \vec{w} = 0$. We write $\vec{v} \perp \vec{w}$.

It usually take three pieces of information to describe a plane in 3-space (3 points in the plane, or a point and the x- and y-slopes), but, using dot products, we can describe a plane using only *two*:

Every plane has a *normal vector* $\vec{n}$; $\vec{n}$ is orthogonal to the vector $\overrightarrow{PQ}$ for any pair of points P and Q in the plane. For example, the vector $\vec{k}$ is perpendicular to the xy-plane, sine it is perpendicular to every vector of the form $(a, b, 0)$. Given such a normal vector $\vec{n}$ and a point (x_0, y_0, z_0) *in* the plane, every other point in the plane must satisfy $\vec{n} \bullet (x - x_0, y - y_0, z - z_0) = 0$; writing this in coordinates gives the equation for the plane;

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0, \text{ where } \vec{n} = (a, b, c)$$

This means that if we are *given* the equation of the plane, we can quickly read off what the normal vector for the plane is, as well.

Another application: projecting one vector onto another.

The idea is to figure out how much of one vector $\vec{v}$ points *in* the direction of another vector $\vec{w}$. The dot product measures to what extent they are pointing in the same direction, so it is only natural that it plays a role.

What we wish to do is to write $\vec{v} = c\vec{w} + \vec{u}$, where $\vec{u} \perp \vec{w}$ (i.e., write $\vec{v}$ as the part pointing in the direction of $\vec{w}$ and the part $\perp \vec{w}$). By solving the equation $(\vec{v} - c\vec{w}) \bullet \vec{w} = 0$, we find that $c = (\vec{v} \bullet \vec{w})/(\vec{w} \bullet \vec{w})$.

We write $c\vec{w} = \text{proj}_{\vec{w}}\vec{v} = \dfrac{\vec{v} \bullet \vec{w}}{\vec{w} \bullet \vec{w}}\vec{w} = \dfrac{\vec{v} \bullet \vec{w}}{||\vec{w}||}\dfrac{\vec{w}}{||\vec{w}||}$ = (orthogonal) projection of $\vec{v}$ onto $\vec{w}$

$\vec{u} = \vec{v} - c\vec{w}$!

The cross product

We saw how to use the dot product to give an equation for a plane, using a normal vector for the plane. But how do we find the normal vector, from the usual pieces of information we will know about the plane? One answer is given by the cross product.

If we set out to find , given vectors for $\vec{v} = (v_1, v_2, v_3)$ and $\vec{w} = (w_1, w_2, w_3)$, a vector $\vec{n} = (a, b, c)$ with $\vec{n} \perp \vec{v}$ and $\vec{n} \perp \vec{w}$, then writing down the equations $\vec{n} \bullet \vec{v} = 0$ and $\vec{n} \bullet \vec{w} = 0$ and solving them will, by making the least effort possible, lead us to the values

$$\vec{n} = \vec{v} \times \vec{w} = (v_2 w_3 - v_3 w_2, -(v_1 w_3 - v_3 w_1), v_1 w_2 - v_2 w_1)$$
$$= (v_2 w_3 - v_3 w_2)\vec{i} - (v_1 w_3 - v_3 w_1)\vec{j} + (v_1 w_2 - v_2 w_1)\vec{k}$$

This vector, which we call the *cross product* of $\vec{v}$ and $\vec{w}$, and denote by $\vec{v} \times \vec{w}$, is perpendicular to both $\vec{v}$ and *vecw*. <u>In addition</u>, its length

$||\vec{v} \times \vec{w}|| = ||\vec{v}|| \cdot ||\vec{w}|| \cdot \sin(\theta)$ = the area of the parallelogram with sides $\vec{v}$ and $\vec{w}$ (where θ is the angle between the vectors). [This fact will be useful to us later on, when we start integration!]

How do you remember this formula? Most people remember it using the notation

$$\vec{v}\times\vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix}\vec{i} - \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix}\vec{j} + \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix}\vec{k}$$

where $\begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} = v_2w_3 - v_3w_2$, etc. The idea is that the first coordinate of the cross product does <u>not</u> use the first coordinates of the two vectors, a pattern which holds for each coordinate. We also multiply by alternating signs $(+,-,+)$.

The cross product satisfies the following equalities:

$$\vec{v}\times\vec{w} = -\vec{w}\times\vec{v} \qquad\qquad (k\vec{v})\times\vec{w} = k(\vec{v}\times\vec{w})$$

One immediate application is to finding the equation for a plane:
To find the plane through the three points P, Q, and R in 3-space, look at the vectors $\vec{v} = \overrightarrow{PQ}$ and $\vec{w} = \overrightarrow{PR}$. These are vectors between points in our plane, and so they give a pair of directions in the plane. They then must both be perpendicular to the normal vector $\vec{n}$ for the plane. But we know that they are both perpendicular to $\vec{v}\times\vec{w}$, and so $\vec{v}\times\vec{w}$ must be perpendicular to the plane. In other words, we can choose our normal vector to *be* $\vec{v}\times\vec{w}$. Using one of our original points (P, say) as a point in the plane, we can write down the equation for the plane using our dot product equation, above.

Differentiation: Partial derivatives

In one-variable calculus, the derivative of a function $y = f(x)$ is defined as the limit of difference quotients:

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

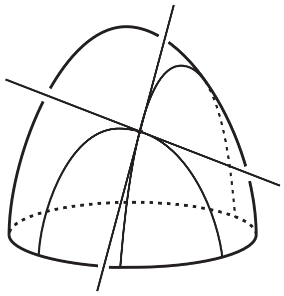
and interpreted as an instantaneous rate of change, or slope of a tangent line.

But a function of two variables has *two variables*; which one do you increment by h to get your difference quotient? The answer is **both of them**, one at a time. In other words, a function of two variables $z = f(x,y)$ has *two* (partial) derivatives:

$$\frac{\partial f}{\partial x} = \lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h} \quad\text{and}\quad \frac{\partial f}{\partial y} = \lim_{h\to 0}\frac{f(x,y+h)-f(x,y)}{h}$$

Essentially, $\dfrac{\partial f}{\partial x}$ is the derivative of f, thought of solely as a function of x (i.e., pretending that y is a constant), while $\dfrac{\partial f}{\partial y}$ is the derivative of f, thought of solely as a function of y.

Different viewpoint, same result:
For one variable calculus $f'(x)$ is the slope of the tangent line to the graph of f. The graph of a (nice!) function of two variables has something we would naturally call a tangent *plane*, and one way to describe a plane is by computing its x- and y-slopes, i.e, the rate of change of f solely in the x- and y- directions. But this is precisely what the limits above calculate; so $\dfrac{\partial f}{\partial x}$ will be the x-slope of the tangent plane, and $\dfrac{\partial f}{\partial y}$ will be the y-slope.



Just as with one variable, there are lots of different notations for describing the partial derivatives: for $z = f(x,y)$,

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(f) = \frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(z) = D_x(f) = D_x(z) = f_x = z_x$$

The algebra of partial derivatives

The basic idea is that since a partial derivative is 'really' the derivative of a function of one *variable* (the other 'variable' is *really* a constant), all of our usual differentiation rules can be applied. so, e.g,

$$\frac{\partial}{\partial x}(f+g)=\frac{\partial f}{\partial x}+\frac{\partial g}{\partial x} \qquad \frac{\partial}{\partial y}(f+g)=\frac{\partial f}{\partial y}+\frac{\partial g}{\partial y}$$
$$\frac{\partial}{\partial x}(c\cdot f)=c\frac{\partial f}{\partial x} \qquad \frac{\partial}{\partial y}(c\cdot f)=c\frac{\partial f}{\partial y}$$
$$\frac{\partial}{\partial x}(f\cdot g)=\frac{\partial f}{\partial x}g+f\frac{\partial g}{\partial x} \qquad \text{(etc.)}$$
$$\frac{\partial}{\partial x}(f/g)=(\frac{\partial f}{\partial x}g-f\frac{\partial g}{\partial x})/g^2 \qquad \text{(etc.)}$$
$$\frac{\partial}{\partial x}(h(f(x,y)))=h'(f(x,y))\cdot\frac{\partial f}{\partial x} \qquad \text{(etc.)}$$

In the end the way we should get used to taking a partial derivative is exactly the same as for functions of one variable; just read from the outside in, applying each rule as it is appropriate. The *only* difference now is that when taking a derivative of a function $z=f(x,y)$, we need to remember that

$$\frac{\partial}{\partial x}(y)=0 \qquad \text{and} \qquad \frac{\partial}{\partial y}(x)=0$$

Tangent planes

In one-variable calculus, we can convince ourselves that a function has a tangent line at a point by zooming in on that point of the graph; the closer we look, the 'straighter' the graph appears to be. At extreme magnification, the graph looks just like a line - its tangent line.

Functions of two variables are really no different; as we zoom in, the graph of our function f starts to look like a plane - the graph's *tangent plane*. Finding the equation of this tangent plane is really a matter of determining its x- and y-slopes, which is precisely what the partial derivatives of f do. The x-slope is the rate of change of the function in the x-direction, i.e., the partial derivative with respect to x; and similarly for the y-slope.

So the equation for the tangent plane to the graph of $z=f(x,y)$ at the point

$$(a,b,f(a,b)) \text{ is} \qquad z=\frac{\partial f}{\partial x}(a,b)(x-a)+\frac{\partial f}{\partial y}(a,b)(y-b)+f(a,b)$$

And just as with one-variable calculus, one use we put this to is to find good approximations to $f(x,y)$ at points near (a,b);

$$f(x,y)\approx\frac{\partial f}{\partial x}(a,b)(x-a)+\frac{\partial f}{\partial y}(a,b)(y-b)+f(a,b), \qquad \text{for } (x,y) \text{ near } (a,b)$$

As with one variable, this also goes hand-in-hand with the idea of *differentials*:

$$df = f_x(a,b)dx + f_y(a,b)dy = \text{differential of } f \text{ at } (a,b)$$

And as before, $f(x,y)-f(a,b)\approx df$, when $dx=x-a$ and $dy=y-b$ are small.

The gradient

$\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ measure the instantaneous rate of change of f in the x- and y-directions, respectively. But what if we want to know the rate of change of f in the direction of the vector $3\vec{\imath}-4\vec{\jmath}$? By thinking of the partial derivatives in a slightly different way, we can get a clue to how to answer this question. By writing $f_x(a,b)=\lim\limits_{h\to 0}\frac{f((a,b)+h(1,0))-f(a,b)}{h}$ and $f_y(a,b)=\lim\limits_{h\to 0}\frac{f((a,b)+h(0,1))-f(a,b)}{h}$, we can make the two derivatives *look* the same; which motivates us to define the *directional derivative* of f at (a,b), in the direction of the vector $\vec{u}$, as

$$f_{\vec{u}}(a,b)=D_{\vec{u}}(a,b)=\lim_{h\to 0}\frac{f((a,b)+h\vec{u})-f(a,b)}{h}$$

[Technically, we need $\vec{u}$ to be a unit vector, $||\vec{u}||$; for other vectors $\vec{v}$, we would define $D_{\vec{v}}(f)=D_{\vec{v}/||\vec{v}||}(f)$.]

But running to the limit definition all of the time would take up way too much of our time; we need a better way to calculate directional derivatives! We can figure out how to do this using differentials:

For $\vec{u} = (u_1, u_2)$, $f((a,b) + h\vec{u} \approx df = f_x(a,b)hu_1 + f_y(a,b)hu_2$, so
$$\frac{f((a,b)+h\vec{u}) - f(a,b)}{h} \approx f_x(a,b)u_1 + f_y(a,b)u_2$$
and so taking the limit, we find that $f_{\vec{u}}(a,b) = f_x(a,b)u_1 + f_y(a,b)u_2 = (f_x(a,b), f_y(a,b)) \bullet \vec{u}$.

The vector $(f_x(a,b), f_y(a,b))$ is going to come up often enough that we will give it its own name;
$$(f_x(a,b), f_y(a,b)) = \nabla(f)(a,b) = \text{grad}(f)(a,b) = \text{the } \textit{gradient} \text{ of } f$$

So the derivative f in the direction of $\vec{u}$ is the dot product of $\vec{u}$ with the gradient of f. This means that (when θ is the angle between ∇f and $\vec{u}$), $D_{\vec{u}}(f) = ||Delf|| \cdot ||\vec{u}|| \cdot \cos(\theta) = ||\nabla f|| \cos(\theta)$. This is the largest when $\cos(\theta) = 1$, i.e., $\theta = 0$ i.e., $\vec{u}$ points in the same direction as ∇f . So ∇f points in the direction of largest increase for the function f, at every point (a,b) . Its length is this maximum rate of increase.

On the other hand, when $\vec{u}$ points in the same direction as the level curve for the point (a,b) (i.e., it is tangent to the level curve), then the rate of change of f in that direction is 0; so $\nabla f \bullet \vec{u} = 0$, i.e, $\nabla f \perp \vec{u}$. This means that ∇f is perpendicular to the level curves of f, at every point (a,b).

Gradients for functions of 3 variables

For functions of 3 variables, everything works pretty much the same. We can make a similar construction of the directional derivative of $w = f(x,y,z)$; using the differential of f,
$$df = f_x(a,b)dx + f_y(a,b)dy + f_z(a,b)dz$$
we can compute that $D_{\vec{u}}(f) = \nabla f \bullet \vec{u}$, where $\nabla f = (f_x, f_y, f_z)$ is the gradient of f. For the exact same reasons, this means that ∇f points in the direction of maximal increase for f, and ∇f is perpendicular to the *level surfaces* for f.

We can use the gradient of functions of 3 variables to help us understand the graphs of functions of two variables, since we can think of the graph of a function of two variables, $z = f(x,y)$, as a *level curve* of a function of 3 variables $g(x,y,z) = f(x,y) - z = 0$.

The gradient of g is perpendicular to its level curves, so it is perpendicular to the graph of f, so gives us the normal vector for the tangent plane to the graph of f. Computing, we find that
$$\nabla g = (f_x, f_y, -1) = \vec{n}$$
so the equation for the tangent plane to the graph of $z = f(x,y)$ at the point $(a, b, f(a,b))$ is
$$(f_x(a,b), f_y(a,b), -1) \bullet (x-a, y-b, z-f(a,b)) = 0$$

The Chain Rule

If f is a function of the variables x and y, and both x and y depend on a single variable t, then in a certain sense, f is a function of t; $f(x,y) = f(x(t), y(t))$; it is a *composition.* To find its derivative *with respect to t*, we can turn to differentials:

$df = f_x dx + f_y dy$, while $dx = \dfrac{dx}{dt}dt$ and $dy = \dfrac{dy}{dt}dt$. Putting these together we get

$df = (\dfrac{\partial f}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dt})dt = \dfrac{df}{dt}dt$, which implies that $\dfrac{df}{dt} = \dfrac{\partial f}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dt}$

This is the (or rather, one of the) Chain Rule(s) for functions of several variables. A similar line of reasoning would lead us to:

If $z = f(u,v)$ and $u = u(x,y)$ and $v = v(x,y)$, then

$\dfrac{\partial f}{\partial x} = \dfrac{\partial f}{\partial u}\dfrac{\partial u}{\partial x} + \dfrac{\partial f}{\partial v}\dfrac{\partial v}{\partial x}$. A similar formula would hold for $\dfrac{\partial f}{\partial y}$.

In general, we can imagine a composition of functions of several variables as a picture with each variable linked by a line going up to functions it is a variable of, and linked by a line going down to variables it is a *function* of, with the original function f at the top. To find the derivative of f with respect to a variable, one finds all paths leading down from f to the variable, multiplying together all of the partial derivatives of one varaible w.r.t. the variable below it, and adding these products together, one for each path. This can, as before, be verified using differentials.

Second Order Partial Derivatives

Just as in one variable calculus, a (partial) derivative is a function; so it has its own partial derivatives. These are called *second partial derivatives.*

We write $\frac{\partial}{\partial x}(\frac{\partial f}{\partial x}) = \frac{\partial^2}{\partial x^2}(\mathrm{f}) = \frac{\partial^2 f}{\partial x^2} = f_{xx} = (f_x)_x$, and similarly for y, and

$\frac{\partial}{\partial y}(\frac{\partial f}{\partial x}) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2}{\partial y \partial x}(\mathrm{f}) = (f_x)_y = f_{xy}$, and similarly for $\frac{\partial^2}{\partial x \partial y} = \frac{\partial}{\partial x}(\frac{\partial f}{\partial y})$

(these are called the *mixed partial derivatives.*

This leads to the slightly confusing convention that $\frac{\partial^2 f}{\partial x \partial y} = f_{yx}$ while $\frac{\partial^2 f}{\partial y \partial x} = f_{xy}$, but as luck would have it:

Fact: If f_{xy} and f_{yx} are both continuous, then they are equal [[Mixed partials are equal.]] So while at first glance a function of two variables would seem to have four second partials, it 'really' has only three. (Similarly, a function of three variables 'really' has six second partials, and not nine.)

In one-variable calculus, the second derivative measures concavity, or the rate at which the graph of f bends. The second partials f_{xx} and f_{yy} measure the bending of the graph of f in the x- and y-directions, while f_{xy} measures the rate at which the x-slope of f changes as you move in the y-direction, i.e., the amount that the graph is *twisting* as you walk in the y direction. The statement that $f_{xy} = f_{yx}$ then says that the amount of twisting in the y-direction is *always* the *same* as the amount of twisting in the x-direction, at any point, which is by no means obvious!

Taylor Approximations

In some sense, the culmination of one-variable calculus is the observation that any function can be approximated by a polynomial; and the polynomial of degree n that 'best' approximates f near the point a is the one which has the same (higher) derivatives as f at a, up to the nth derivative. This leads to the definition of the *Taylor polynomial* :

$$p_n(x) = f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Functions of two variables are not much different; we just replace the word 'derivative' with '*partial* derivative'! So for example, the degree one Taylor polynomial is

$$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

which is nothing more than our old formula for the tangent plane to the graph of f at the point $(a, b, f(a,b))$.

We will soon need the second degree version (which for simplicity we will write for the point (a, b) = (0,0)) :

$$Q(x,y) = L(x,y) + \frac{f_{xx}(0,0)}{2}x^2 + f_{xy}(0,0)xy + \frac{f_{yy}(0,0)}{2}y^2 = L(x,y) + Ax^2 + Bxy + Cy^2$$

As before, L and Q are the 'best' linear and quadratic approximations to f, near the point (a, b), in a sense that can be made precise; basically, $L - f$ shrinks to 0 like a quadratic, near (a, b), while $Q - f$ shrinks like a cubic (which shrinks to 0 *faster*, when your input is small).

Differentiability

In one-variable calculus, 'f is differentiable' is just another way of saying 'the derivative of f exists'. But with several variables, differentiablility means **more** than that all of the partial derivatives exist.

A function of several variables is *differentiable* at a point if the tangent plane to the graph of f at that point makes a good approximation to the function, near the point of tangency. In the words of the previous paragraph, $f(x,y) - L(x,y)$ shrinks to 0 *faster* than a linear function would.

A basic fact, that we keep using, is that if the partial derivatives of f don't just *exist* at a point, but are also **continuous** near the point, then f is differentiable in this more precise sense.

Local Extrema

The partial derivatives of f measuire the rate of change of f in each of the coordinate directions. So they are giving us partial information (no pun intended) about how thew function f is rising and falling. And just as in one-variable calculus, we ought to be able to turn this into a procedure for findong out when a function is at its maximum or minimum.

The basic idea is that at a max or min for f, then, thinking of f just as a function of x, we would *still* think we were at a max or min, so the derivative, as a function of x, will be 0 (if it is defined). In other words, $f_x = 0$. similarly, we would find that $f_y = 0$, as well. following one-variable theory, therefore, we say that

A point (a, b) is a **critical point** for the function f if $f_x(a, b)$ and $f_y(a, b)$ are *each* either 0 or undefined. (A similar notion would hold for functions of more than two variables.)

Just as with the one-variable theory, then, if we wish to find the max or min of a function, what we first do is find the critical points; *if* the function has a max or min, it will occur at a critical point.

And just as before, we have a 'Second Derivative Test' for figuring out the difference between a (local) max and a (local) min (or *neither*, which we will call a *saddle point*). The point is that at a critical point, f looks like its quadratic approximation, which (simplifying things somewhat) is described as $Q(x, y) = Dx^2 + Exy + Fy^2$ (since the first derivatives are 0). By completing the square, we can see that the actual shape of the graph of Q is basically described by *one number*, called the descriminant, which (in terms of partial derivatives) is given by

$$D = f_{xx}(\text{a,b})f_{yy}(\text{a,b}) - (f_{xy}(\text{a,b}))^2$$

(Basically, Q looks like one of $x^2 + y^2$ (local min), $-x^2 - y^2$ (local max), or $x^2 - y^2$ (saddle), and D tells you if the signs are the same ($D > 0$) or opposite ($D < 0$) . More specifically, if, at a critical point (a, b),

$D > 0$ and $f_{xx} > 0$ then (a, b) is a local min; if

$D > 0$ and $f_{xx} < 0$ then (a, b) is a local max; and if

$D < 0$, then (a, b) is a saddle point

(We get no information if $D = 0$.)