Math 208H, Section 1

Practice problems for Exam 1 (Solutions)

1. Find the **sine** of the angle between the vectors (1,−1,2) and (1,2,1) .

We can use the dot product (dividing by lengths) to compute the cosine of the angle, and then from that the sine. Or we can use $|\vec{u}\times\vec{v}| = |\vec{u}||\vec{v}|\sin(\theta)$ to compute the sine, by finding the cross product and computing lengths.

$\sin(\theta) = \sqrt{(-5)^2+1^2+3^2}/(\sqrt{1^2+(-1)^2+2^2}\cdot\sqrt{1^2+2^2+1^2}) = \sqrt{35}/(\sqrt{6}\cdot\sqrt{6}) = \sqrt{35}/6$

This is consistent with $\cos(\theta) = (1\cdot 1+(-1)\cdot 2+2\cdot 1)/(\sqrt{6}\cdot\sqrt{6}) = 1/6$.

2. Find a vector of length 3 that is perpendicular to both

$$\vec{v} = \langle 1,3,5\rangle \text{ and } \vec{w} = \langle 2,1,-1\rangle \ .$$

A vector perpendicular to both is given by the cross product, so we compute

$$\vec{v}\times\vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 3 & 5 \\ 2 & 1 & -1 \end{vmatrix} = \begin{vmatrix} 3 & 5 \\ 1 & -1 \end{vmatrix}\vec{i} - \begin{vmatrix} 1 & 5 \\ 2 & -1 \end{vmatrix}\vec{j} + \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix}\vec{k}$$

$= \langle -3-5, -(-1-10), 1-6\rangle = \langle -8,11,-5\rangle$

[We can test that this is perpendicular to the two vectors by computing dot products...]

This vector has length $\sqrt{64+121+25} = \sqrt{210}$; since we want a vector of length 3, we take the appropriate scalar multiple:

$\vec{N} = \dfrac{3}{\sqrt{210}}\langle -8,11,-5\rangle$ has length 3 and is $\perp$ tp $\vec{v}$ and $\vec{w}$. [Its negative also works...]

3. Show that if the vectors $\vec{\mathbf{v}}=(a_1,a_2,a_3)$ and $\vec{\mathbf{w}}=(b_1,b_2,b_3)$ have the same length, then the vectors $\vec{\mathbf{v}}+\vec{\mathbf{w}}$ and $\vec{\mathbf{v}}-\vec{\mathbf{w}}$ are perpendicular to one another.

We wish to know that $(\vec{\mathbf{v}}+\vec{\mathbf{w}})\circ(\vec{\mathbf{v}}-\vec{\mathbf{w}}) = 0$. But expanding this out, we find that it is equal to $\vec{\mathbf{v}}\circ\vec{\mathbf{v}}-\vec{\mathbf{w}}\circ\vec{\mathbf{w}}$. This will be equal to 0 precisely when $|\vec{v}|^2 = \vec{\mathbf{v}}\circ\vec{\mathbf{v}} = \vec{\mathbf{w}}\circ\vec{\mathbf{w}} = |\vec{w}|^2$. This in turn, means that $\vec{\mathbf{v}}$ and $\vec{\mathbf{w}}$ have the same length.

4. Find the equation of the plane in 3-space which passes through the three points $(1,2,1)$, $(6,1,2)$, and $(9,-2,1)$. Does the point $(3,2,1)$ lie on this plane?

To find the equation, we need a point and a normal vector; the normal can be found by a cross product. $\vec{N} = \vec{PQ}\times\vec{PR} = (5,-1,1)\times(8,-4,0) = (4,8,-12)$. Then the equation is $(4,8,-12)\circ(x-1,y-2,z-1) = 0$, or $4x+8y-12z = 8$ (or $x+2y-3z = 2$ (!)). [Check: the 3 points satisfy the equation!] Checking, $4\cdot 3+8\cdot 2-12\cdot 1 = 16 \neq 8$, so the point does not lie on the plane.

5. Find the partial derivatives of the following functions:

(a) $f(x,y,z) = x\tan(2x+yz)$

We have $f_x = \tan(2x+yz)+x\sec^2(2x+yz)\cdot 2$, $f_y = x\sec^2(2x+yz)\cdot z$, and $f_z = x\sec^2(2x+yz)\cdot y$.

(b) $g(x,y) = \dfrac{x^2y-ty^4}{\sin(3y)+4}$ We have $g_x = \dfrac{(2xy)(\sin(3y)+4)-(x^2y-ty^4)(0)}{(\sin(3y)+4)^2}$, and $g_y = \dfrac{(x^2 - 4ty^3)(\sin(3y)+4) - (x^2y - ty^4)(3\cos(3y))}{(\sin(3y)+4)^2}$. Since the question didn't ask us to do anything with these, why simplify them?

6. Find the equation of the tangent plane to the graph of the equation $f(x,y,z) = xy^2 + x^2z - xyz = 5$, at the point $(-1,1,3)$.

$f_x = y^2 + 2xz - yz$, $f_y = 2xy + x^2 - xz$, and $f_z = x^2 - xy$. The normal vector to the plane will be $(f_x(-1,1,3), f_y(-1,1,3), f_z(-1,1,3)) = (1-6-3, -2+1+3, 1+1) = (-8,2,2)$. Together with the point of tangency, this gives us the equation
$-8(x-(-1)) + 2(y-1) + 2(z-3) = 0$, or $-8x + 2y + 2z = 16$, or $4x - y - z = -8$.

7. Calculate the first and second partial derivatives of the function $h(x,y) = \dfrac{\sin(x+y)}{y}$

It may help a bit to write this function as $h(x,y) = y^{-1}\sin(x+y)$. Then we have
$h_x = y^{-1}\cos(x+y)\cdot 1 = y^{-1}\cos(x+y)$,
$h_y = -y^{-2}\sin(x+y) + y^{-1}\cos(x+y)\cdot 1 = -y^{-2}\sin(x+y) + y^{-1}\cos(x+y)$. Then
$h_{xx} = (h_x)_x = y^{-1}(-\sin(x+y)\cdot 1) = -y^{-1}\sin(x+y)$
$h_{yx} = h_{xy} = (h_x)_y = -y^{-2}\cos(x+y) + y^{-1}(-\sin(x+y)\cdot 1)$
$= -y^{-2}\cos(x+y) - y^{-1}\sin(x+y)$
$h_{yy} = (h_y)_y$
$= [2y^{-3}\sin(x+y) - y^{-2}(\cos(x+y)\cdot 1)] + [-y^{-2}\cos(x+y) + y^{-1}(-\sin(x+y)\cdot 1)]$
Again, we don't want to do anything with it, so why bother simplifying it...

8. In which direction is the function $f(x,y) = x^4y - 3x^2y^2$ increasing the fastest, at the point (1,2) ? In which directions is the function *neither* increasing *nor* decreasing?

f increases fastest in the direction of the gradient, so we compute:
$\nabla f = (4x^3y - 6xy^2, x^4 - 6x^2y)$, which at (1,2) gives $\vec{v} = (8-24, 1-12) = (-16,-11)$. This is the drection of fastest increase (you can divide by its length if you want a unit vector...).

For no increase/decrease, what we want is $D_{\vec{w}}f = \nabla f \circ \vec{w} = 0$, so we want $(-16,11)\circ(\alpha,\beta) = -16\alpha - 11\beta = 0$; we can do this, for example, with $\vec{w} = (\alpha,\beta) = (11,-16)$. [There are many other answers, all scalar multiples of this one.]

9. If $f(x,y) = x^2y^5 - x + 3y - 4$, $x = x(u,v) = \dfrac{u}{u+v}$ and $y = y(u,v) = uv - u$, use the Chain Rule to find $\dfrac{\partial f}{\partial u}$ when $u = 1$ and v=0 .

First, when $(u,v) = (1,0)$, then $x = 1/(1+0) = 1$ and $y = 1\cdot 0 - 1 = -1$. From the chain rule, we know that $f_u = f_x x_u + f_y y_u$, evaluated at $(x,y) = (1,-1)$ and $(u,v) = (1,0)$. We compute:

$f_x = 2xy^5 - 1 = -2-1 = -3$, $f_y = 5x^2y^4 + 3 = 5+3 = 8$, $x_u = \dfrac{(1)(u+v)-(u)(1)}{(u+v)^2} = \dfrac{v}{(u+v)^2} = 0$, and $y_u = v - 1 = 0 - 1 = -1$; so at $(u,v) = (1,0)$ we have $f_u(1,0) = (-3)(0) + (8)(-1) = -8$.

10. If $f(x,y) = \dfrac{x^2y}{x+y}$, and $\gamma(t) = (x(t), y(t))$ is a parametrized curve in the domain of f with $\gamma(0) = (2,-1)$ and $\gamma'(0) = (3,5)$, then what is $\dfrac{d}{dt} f(\gamma(t))\Big|_{t=0}$?

By the chain rule, $\dfrac{df}{dt} = f_x x_t + f_y y_t$. We compute: $f_x = \dfrac{(2xy)(x+y) - (x^2y)(1)}{(x+y)^2}$ and $f_y = \dfrac{(x^2)(x+y) - (x^2y)(1)}{(x+y)^2}$.

At $(2,-1)$, these are $f_x = \dfrac{(-4)(1) - (-4)(1)}{(1)^2} = 0$ and $f_y = \dfrac{(4)(1) - (-4)(1)}{(1)^2} = 8$, so $\dfrac{df}{dt} = f_x x_t + f_y y_t = (0)(3) + (8)(5) = 40$.

11. Find the **second** partial derivatives of the function $h(x,y) = x\sin(xy^2)$.

We compute: $h_x = (1)(\sin(xy^2)) + (x)(\cos(xy^2))(y^2) = \sin(xy^2) + xy^2\cos(xy^2)$
$h_y = x(\cos(xy^2))(2xy) = 2x^2y\cos(xy^2)$. Then for the second partials:
$h_{xx} - (h_x)_x = (\cos(xy^2))(y^2) + [(y^2)(\cos(xy^2)) + (xy^2)(-\sin(xy^2))(y^2)]$
$= 2y^2\cos(xy^2) - xy^4\sin(xy^2)$
$h_{xy} = h_{yx} = (h_y)_x = (4xy)(\cos(xy^2)) + (2x^2y)(-\sin(xy^2))(y^2)$
$= 4xy\cos(xy^2) - 2x^2y^3\sin(xy^2)$
$h_{yy} = (h_y)_y = (2x^2)(\cos(xy^2)) + (2x^2y)(-\sin(xy^2))(2xy)$
$= 2x^2\cos(xy^2) - 4x^3y^2\sin(xy^2)$

12. For which value(s) of c are the vectors $\vec{v} = (1,2,c)$ and $\vec{w} = (-5,2c,4)$ orthogonal?

We want $\vec{v} \bullet \vec{w} = 0$, so $0 = (1,2,c) \bullet (-5,2c,4) = -5 + 4c + 4c = -5 + 8c$, so $8c = 5$ and so $c = 5/8$. This gives the vectors

$$\vec{v} = (1,2,\frac{5}{8}) \text{ and } \vec{w} = (-5,\frac{5}{4},4) .$$

[As a check, $\vec{v} \bullet \vec{w} = (1,2,\frac{5}{8}) \bullet (-5,\frac{5}{4},4) = -5 + \frac{5}{2} + \frac{5}{2} = 0$, as desired.]

13. Find the equation of the plane passing through the points

$$(2,3,5)\ ,\ (1,-1,0)\ ,\ \text{and } (1,1,2) .$$

Labeling the points P, Q, and R for convenience, we have $\vec{v} = \vec{PQ} = (-1,-4,-5)$ and $\vec{w} = \vec{PR} = (-1,-2,-3)$. These are directions in the plane, and so their cross product will be normal to the plane. So we compute

$$\vec{n} = \vec{v}\times\vec{w} = (\begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix}, -\begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix}, \begin{vmatrix} v_1 & v_2 \\ w_1 & w_1 \end{vmatrix}) = (\begin{vmatrix} -4 & -5 \\ -2 & -3 \end{vmatrix}, -\begin{vmatrix} -1 & -5 \\ -1 & -3 \end{vmatrix}, \begin{vmatrix} -1 & -4 \\ -1 & -2 \end{vmatrix})$$
$= (12 - 10, -(3-5), 2-4) = (2,2,-2)$.

[As a check, we can compute $\vec{v} \bullet \vec{n} = -2 - 8 + 10 = 0$ and $\vec{w} \bullet \vec{n} = -2 - 4 + 6 = 0$.]

With a normal $\vec{n} = (2,2,-2)$ to the plane and a point $P = (2,3,5)$ on the plane, we can give the equation for the plane as $\vec{n} \bullet [(x,y,z) - (2,3,5)] = 0$, i.e.,

$$2(x-2)+2(y-3)-2(z-5)=0 \ .$$

[There are, of course, many other equivalent answers, obtained by choosing, for example, another point to use as the tails of our vectors....]

14. What is the rate of change of the function $f(x,y) = \dfrac{xy}{x+2y}$, at the point $(4,2)$, and in the direction of the vector $\vec{v} = (1,1)$?

The rate of change is the directional derivative, computed as $\nabla f(4,2) \bullet \vec{v}$. So we compute:

$f(x,y) = xy(x+2y)^{-1}$, so $f_x = (y)(x+2y)^{-1} + (xy)[(-1)(x+2y)^{-2}(1)]$, and
$f_y = (x)(x+2y)^{-1} + (xy)[(-1)(x+2y)^{-2}(2)]$. So

$\nabla f(4,2) = ((2)(8)^{-1} + (8)(-1)(8)^{-2}, (4)(8)^{-1} + (8)(-1)(8)^{-2}(2)) = (\frac{1}{4} - \frac{1}{8}, \frac{1}{2} - \frac{1}{4}) = (\frac{1}{8}, \frac{1}{4})$

So the rate of change is $\nabla f(4,2) \bullet \vec{v} = (\frac{1}{8}, \frac{1}{4}) \bullet (1,1) = \frac{1}{8} + \frac{1}{4} = \frac{3}{8}$.

[Under some interpretations, we should divide this number by $||(1,1)|| = \sqrt{2}$, in order to be using the <u>unit</u> vector pointing in the direction of $\vec{v}$.]

15. Find the equation of the plane tangent to the graph of the function
$g(x,y) = x^3y - 4x^2y^2 + 2xy^4$ at the point $(2,1,g(2,1))$.

We can describe this plane using a point on the plane and its $x-$ and $y-$slopes, all of which the function can provide.

The point of tangency $(2,1,g(2,1)) = (2,1,8-16+4) = (2,1,-4)$ is a point on the plane.

For the slopes, we compute:
$g_x = 3x^2y - 8xy^2 + 2y^4$, so $f_x(2,1) = 12 - 16 + 2 = -2 = m = x-$slope.
$g_y = x^3 - 8x^2y + 8xy^3$, so $f_y(2,1) = 8 - 32 + 16 = -8 = n = y-$slope.

So the equation for the tangent plane is given by

$$z = g(2,1) + g_x(2,1)(x-2) + g_y(2,1)(y-1) = -4 - 2(x-2) - 8(y-1)$$

Multiplying out, this can be converted to $z = -4 - 2x + 4 - 8y + 8 = -2x - 8y + 8$.

16. If $x = u^2v$ and $y = uv^2$, then show how to express the partial derivatives of
$g(u,v) = f(x(u,v), y(u,v))$ at the point $(u,v) = (2,-1)$, in terms of the (at the moment unknown) partial derivatives $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$.

Writing $z = g(u,v) = f(x(u,v), y(u,v))$, by the Chain rule, we know that $z_u = z_x x_u + z_y y_u$ and $z_v = z_x x_v + z_y y_v$. We can compute

$x(2,-1) = 2^2(-1) = -4$ and $y(2,-1) = 2(-1)^2 = 2$, so $(x,y) = (-4,2)$, while

$x_u = 2uv$, $x_v = u^2$, $y_u = v^2$, and $y_v = 2uv$, and so at $(u,v) = (2,-1)$, we have

$x_u = -4$, $x_v = 4$, $y_u = 1$, and $y_v = -4$. So at $(u, v) = (2, -1)$, we have
$g_u(2, -1) = [f_x(-4, 2)][-4] + [f_y(-4, 2)][1] = -4f_x(-4, 2) + f_y(-4, 2)$, and
$g_v(2, -1) = [f_x(-4, 2)][4] + [f_y(-4, 2)][-4] = 4f_x(-4, 2) - 4f_y(-4, 2)$.

17. Find the **second** partial derivatives of the function $h(x, y) = xe^{xy}$.

We compute:
$h_x = (1)(e^{xy}) + (x)(e^{xy}y) = e^{xy} + xye^{xy}$ and $h_y = (0)(e^{xy}) + (x)(e^{xy}x) = x^2e^{xy}$. So

$h_{xx} = (h_x)_x = (e^{xy})(y) + [(y)e^{xy} + (xy)(e^{xy}y)] = 2ye^{xy} + xy^2e^{xy}$,
$h_{xy} = (h_x)_y = (e^{xy})(x) + [(x)(e^{xy}) + (xy)(e^{xy}x)] = 2xe^{xy} + x^2ye^{xy}$,
$h_{yx} = (h_y)_x = (2x)(e^{xy}) + (x^2)(e^{xy}y) = 2xe^{xy} + x^2ye^{xy} = h_{xy}$, and
$h_{yy} = (x^2)(e^{xy}x) = x^3e^{xy}$.

18. Find the local extrema of the function $f(x, y) = 2x^4 - 2xy + y^2$, and determine, for each, if it is a local max. local min, or saddle point.

Local extrema occur at critical points, so we compute: $f_x = 8x^3 - 2y$ and $f_y = -2x + 2y$. These are never undefined, so our only critical points will occur when both are 0. $f_y = -2x + 2y = 0$ means $2y = 2x$, so $y = x$. Substituting this into $f_x = 8x^3 - 2y = 0$ gives $8x^3 - 2x = (2x)(4x^2 - 1) = 0$, so either $x = 0$, or $4x^2 - 1 = 0$, so $x = 0$ or $x = 1/2$ or $x = -1/2$. This yields the three critical points $(0, 0)$, $(1/2, 1/2)$, and $(-1/2, -1/2)$.

To determine their character, we need the Hessian: $f_{xx} = 24x^2$, $f_{xy} = -2$, and $f_{yy} = 2$, so $H = f_{xx}f_{yy} - (f_{xy})^2 = 48x^2 - 4$. At $(0, 0)$ $H = -4 < 0$, so $(0, 0)$ is a saddle point. At $(1/2, 1/2)$, $H = 48/4 - 4 = 12 - 4 = 8 > 0$ and $f_{xx} = 24/4 = 6 > 0$, so $(1/2, 1/2)$ is a local min. And at $(-1/2, -1/2)$, $H = 48/4 - 4 = 12 - 4 = 8 > 0$ and $f_{xx} = 24/4 = 6 > 0$ as well, so $(-1/2, -1/2)$ is also a local min.