Math 208H, Section 1

Exam 1 Solutions

1. (15 pts.) For which value(s) of c are the vectors $\vec{v} = (1, 2, c)$ and $\vec{w} = (-5, 2c, 4)$ orthogonal?

We want $\vec{v} \bullet \vec{w} = 0$, so $0 = (1, 2, c) \bullet (-5, 2c, 4) = -5 + 4c + 4c = -5 + 8c$, so $8c = 5$ and so $c = 5/8$. This gives the vectors

$$\vec{v} = (1, 2, \frac{5}{8}) \text{ and } \vec{w} = (-5, \frac{5}{4}, 4) \ .$$

[As a check, $\vec{v} \bullet \vec{w} = (1, 2, \frac{5}{8}) \bullet (-5, \frac{5}{4}, 4) = -5 + \frac{5}{2} + \frac{5}{2} = 0$, as desired.]

2. (20 pts.) Find the equation of the plane passing through the points

$$(2, 3, 5) \ , \ (1, -1, 0) \ , \text{ and } (1, 1, 2) \ .$$

Labeling the points P, Q, and R for convenience, we have $\vec{v} = \vec{PQ} = (-1, -4, -5)$ and $\vec{w} = \vec{PR} = (-1, -2, -3)$. These are directions <u>in</u> the plane, and so their cross product will be normal to the plane. So we compute

$$\vec{n} = \vec{v} \times \vec{w} = (\begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix}, -\begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix}, \begin{vmatrix} v_1 & v_2 \\ w_1 & w_1 \end{vmatrix}) = (\begin{vmatrix} -4 & -5 \\ -2 & -3 \end{vmatrix}, -\begin{vmatrix} -1 & -5 \\ -1 & -3 \end{vmatrix}, \begin{vmatrix} -1 & -4 \\ -1 & -2 \end{vmatrix})$$
$$= (12 - 10, -(3 - 5), 2 - 4) = (2, 2, -2).$$

[As a check, we can compute $\vec{v} \bullet \vec{n} = -2 - 8 + 10 = 0$ and $\vec{w} \bullet \vec{n} = -2 - 4 + 6 = 0$.]

With a normal $\vec{n} = (2, 2, -2)$ to the plane and a point $P = (2, 3, 5)$ on the plane, we can give the equation for the plane as $\vec{n} \bullet [(x, y, z) - (2, 3, 5)] = 0$, i.e.,

$$2(x - 2) + 2(y - 3) - 2(z - 5) = 0 \ .$$

[There are, of course, many other equivalent answers, obtained by choosing, for example, another point to use as the tails of our vectors....]

3. (15 pts.) What is the rate of change of the function $f(x, y) = \dfrac{xy}{x + 2y}$, at the point $(4, 2)$, and in the direction of the vector $\vec{v} = (1, 1)$?

The rate of change is the directional derivative, computed as $\nabla f(4, 2) \bullet \vec{v}$. So we compute:

$f(x, y) = xy(x + 2y)^{-1}$, so $f_x = (y)(x + 2y)^{-1} + (xy)[(-1)(x + 2y)^{-2}(1)]$, and
$f_y = (x)(x + 2y)^{-1} + (xy)[(-1)(x + 2y)^{-2}(2)]$. So

$$\nabla f(4, 2) = ((2)(8)^{-1} + (8)(-1)(8)^{-2}, (4)(8)^{-1} + (8)(-1)(8)^{-2}(2)) = (\frac{1}{4} - \frac{1}{8}, \frac{1}{2} - \frac{1}{4}) = (\frac{1}{8}, \frac{1}{4})$$

So the rate of change is $\nabla f(4, 2) \bullet \vec{v} = (\frac{1}{8}, \frac{1}{4}) \bullet (1, 1) = \frac{1}{8} - \frac{1}{4} = \frac{1}{8}$.

[Under some interpretations, we should divide this number by $||(1,1)|| = \sqrt{2}$, in order to be using the unit vector pointing in the direction of $\vec{v}$.]

4. (15 pts.) Find the equation of the plane tangent to the graph of the function

$g(x,y) = x^3y - 4x^2y^2 + 2xy^4$ at the point $(2,1,g(2,1))$.

We can describe this plane using a point on the plane and its $x-$ and $y-$slopes, all of which the function can provide.

The point of tangency $(2,1,g(2,1)) = (2,1,8-16+4) = (2,1,-4)$ is a point on the plane. For the slopes, we compute:

$g_x = 3x^2y - 8xy^2 + 2y^4$, so $f_x(2,1) = 12 - 16 + 2 = -2 = m = x-$slope.
$g_y = x^3 - 8x^2y + 8xy^3$, so $f_y(2,1) = 8 - 32 + 16 = -8 = n = y-$slope.

So the equation for the tangent plane is given by

$$z = g(2,1) + g_x(2,1)(x-2) + g_y(2,1)(y-1) = -4 - 2(x-2) - 8(y-1)$$

Multiplying out, this can be converted to $z = -4 - 2x + 4 - 8y + 8 = -2x - 8y + 8$.

5. (15 pts.) If $x = u^2v$ and $y = uv^2$, then show how to express the partial derivatives of $g(u,v) = f(x(u,v),y(u,v))$ at the point $(u,v) = (2,-1)$, in terms of the (at the moment unknown) partial derivatives $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$.

Writing $z = g(u,v) = f(x(u,v),y(u,v))$, by the Chain rule, we know that $z_u = z_x x_u + z_y y_u$ and $z_v = z_x x_v + z_y y_v$. We can compute

$x(2,-1) = 2^2(-1) = -4$ and $y(2,-1) = 2(-1)^2 = 2$, so $(x,y) = (-4,2)$, while

$x_u = 2uv$, $x_v = u^2$, $y_u = v^2$, and $y_v = 2uv$, and so at $(u,v) = (2,-1)$, we have

$x_u = -4$, $x_v = 4$, $y_u = 1$, and $y_v = -4$. So at $(u,v) = (2,-1)$, we have

$g_u(2,-1) = [f_x(-4,2)][-4] + [f_y(-4,2)][1] = -4f_x(-4,2) + f_y(-4,2)$, and
$g_v(2,-1) = [f_x(-4,2)][4] + [f_y(-4,2)][-4] = 4f_x(-4,2) - 4f_y(-4,2)$.

6. (20 pts.) Find the **second** partial derivatives of the function $h(x,y) = xe^{xy}$.

We compute:
$h_x = (1)(e^{xy}) + (x)(e^{xy}y) = e^{xy} + xye^{xy}$ and $h_y = (0)(e^{xy}) + (x)(e^{xy}x) = x^2e^{xy}$. So

$h_{xx} = (h_x)_x = (e^{xy})(y) + [(y)e^{xy} + (xy)(e^{xy}y)] = 2ye^{xy} + xy^2e^{xy}$,
$h_{xy} = (h_x)_y = (e^{xy})(x) + [(x)(e^{xy}) + (xy)(e^{xy}x)] = 2xe^{xy} + x^2ye^{xy}$,
$h_{yx} = (h_y)_x = (2x)(e^{xy}) + (x^2)(e^{xy}y) = 2xe^{xy} + x^2ye^{xy} = h_{xy}$, and
$h_{yy} = (x^2)(e^{xy}x) = x^3e^{xy}$.