A random integral

We set up the integral for the surface area of the graph of the function $f(x,y) = e^x e^y$, using the parametrization

$$x = u \ , \ y = v \ , \ z = f(u,v) = e^u e^v$$
$$\text{for } 0 \le u \le 1 \text{ and } 0 \le v \le 2$$

giving us the integral $\displaystyle\int_0^2 \int_0^1 \sqrt{(e^u e^v)^2 + (e^u e^v)^2 + 1}\ du\ dv = \int_0^2 \int_0^1 \sqrt{1 + 2e^{2u}e^{2v}}\ du\ dv$

After noting that $\displaystyle\int \sqrt{1+e^{2x}}\ dx = \int \frac{\sqrt{1+(e^x)^2}}{e^x}\ (e^x\ dx)$ is an integral that we "can" solve for (by a bewildering sequence of substitutions), we wisely left the integral alone.

But Maple 15 can carry out (some) symbolic integrations! In particular, it can, it turns out, quickly do the <u>first</u> of the two integrations:

$$\int \sqrt{1+2e^{2u}e^{2v}}\ du = \sqrt{1+2e^{2u}e^{2v}} + \frac{1}{2}\ln(\sqrt{1+2e^{2u}e^{2v}} - 1) - \frac{1}{2}\ln(\sqrt{1+2e^{2u}e^{2v}} + 1)$$

The fun, of course, comes when (after evaluating this at $u = 1$ and $u = 0$ and subtracting) we try to compute the second integral! Maple, as it happens, can "do" this one, too, giving

$$\begin{aligned}
&\sqrt{3} + \frac{1}{8}[\ln(\sqrt{2e^2+1}+1)]^2 + \frac{1}{2}\ln(\sqrt{2e^2+1}+1) - \frac{1}{8}[\ln(\sqrt{2e^4+1}-1]^2 \\
&-\frac{1}{8}[\ln(\sqrt{2e^2+1}-1]^2 - \frac{1}{2}\mathrm{dilog}(\frac{1}{2}\sqrt{2e^4+1}+\frac{1}{2}) + \frac{1}{2}\ln(2)\ln(\sqrt{2e^2+1}-1) \\
&-\frac{1}{4}\ln(\sqrt{2e^2+1}-1)\ln(\sqrt{2e^2+1}+1) - \frac{1}{2}\mathrm{dilog}(\frac{1}{2}\sqrt{2e^2+1}+\frac{1}{2}) \\
&+\frac{1}{2}\mathrm{dilog}(\frac{1}{2}\sqrt{2e^6+1}+\frac{1}{2}) + \frac{1}{2}\ln(\sqrt{2e^4+1}+1) + \frac{1}{2}\ln(\sqrt{2e^6+1}-1) \\
&+\frac{1}{8}[\ln(\sqrt{2e^4+1}+1)]^2 - \frac{1}{8}[\ln(\sqrt{2e^6+1}+1)]^2 + \frac{1}{2}\ln(\sqrt{3}-1) - \frac{1}{2}\ln(\sqrt{3}+1) \\
&+\frac{1}{8}[\ln(\sqrt{3}-1)]^2 - \frac{1}{8}[\ln(3\sqrt{3}+1)]^2 + \frac{1}{2}\mathrm{dilog}(\frac{1}{2}\sqrt{3}+\frac{1}{2}) - \frac{1}{2}\ln(\sqrt{2e^4+1}-1) \\
&-\frac{1}{2}\ln(2)\ln(\sqrt{3}-1) + \frac{1}{4}\ln(\sqrt{3}-1)\ln(\sqrt{3}+1) - \frac{1}{2}\ln(\sqrt{2e^2+1}-1) \\
&+\frac{1}{2}\ln(2)\ln(\sqrt{2e^4+1}-1) - \frac{1}{4}\ln(\sqrt{2e^4+1}-1)\ln(\sqrt{2e^4+1}+1) - \frac{1}{2}\ln(2)\ln(\sqrt{2e^6+1}-1) \\
&+\frac{1}{4}\ln(\sqrt{2e^6+1}-1)\ln(\sqrt{2e^6+1}+1) - \sqrt{2e^4+1} - \frac{1}{2}\ln(\sqrt{2e^6+1}+1) \\
&-\sqrt{2e^2+1} + \sqrt{2e^6+1} + \frac{1}{8}[\ln(\sqrt{2e^6+1}-1)]^2
\end{aligned}$$

which evaluates numerically to (approximately) 15.71448653 . But note that Maple has "cheated"; it's answer is written in terms of both the logarithm function $\ln(x)$ <u>and</u> the **dilogarithm** function,

$$\mathrm{dilog}(x) = \int_1^x \frac{\ln(t)}{1-t}\ dt$$

This is not an 'elementary' function as we know them, but it arises in many problems; enough, at least, to have been given its own name...