Math 208H, Section 1

Exam 3 Solutions

1. Find the integral of the function $f(x,y,z) = xy$ over the region in 3-space lying in the first octant (i.e., where $x \geq 0$, $y \geq 0$ and $z \geq 0$ and below the plane $x+y+z=1$.

The region R in question can be describe as $0 \leq z \leq 1-x-y$ for (x,y) in the shadow of the region. We will hit the region so long as $1-x-y \geq 0$, so $x+y \leq 1$, so $0 \leq y \leq 1-x$, for $0 \leq x \leq 1$. So our region is

$$0 \leq z \leq 1-x-y \text{ for } 0 \leq y \leq 1-x, \text{ for } 0 \leq xleq1\ .$$

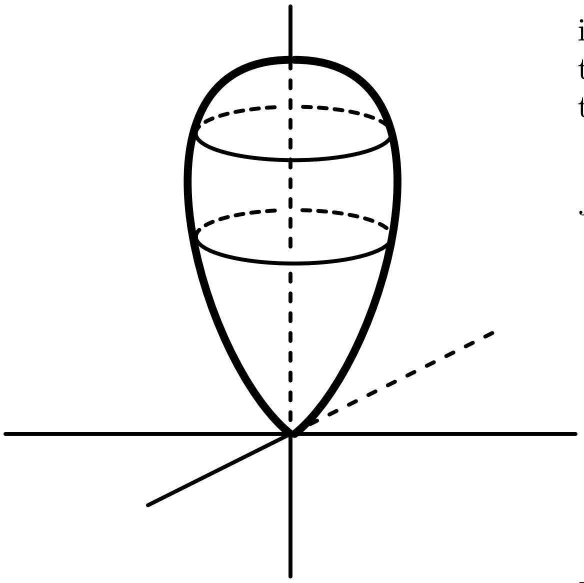
This gives us the integral

$$\int\int\int_R xy\ dV = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} xy\ dz\ dy\ dx = \int_0^1 \int_0^{1-x} xyz\Big|_0^{1-x-y}\ dy\ dx$$
$$= \int_0^1 \int_0^{1-x} xy(1-x-y)\ dy\ dx = \int_0^1 \int_0^{1-x} xy - x^2y - xy^2\ dy\ dx$$
$$= \int_0^1 \frac{1}{2}xy^2 - \frac{1}{2}x^2y^2 - \frac{1}{3}xy^3\Big|_0^{1-x}\ dx = \int_0^1 \frac{1}{2}x(1-x)^2 - \frac{1}{2}x^2(1-x)^2 - \frac{1}{3}x(1-x)^3\ dx$$
$$= \int_0^1 \frac{1}{2}x(1-2x+x^2) - \frac{1}{2}x^2(1-2x+x^2) - \frac{1}{3}x(1-3x+3x^2-x^3)\ dx$$
$$= \frac{1}{6}\int_0^1 3x(1-2x+x^2) - 3x^2(1-2x+x^2) - 2x(1-3x+3x^2-x^3)\ dx$$
$$= \frac{1}{6}\int_0^1 3x - 6x^2 + 3x^3 - 3x^2 + 6x^3 - 3x^4 - 2x + 6x^2 - 6x^3 + 2x^4\ dx$$
$$= \frac{1}{6}\int_0^1 x - 3x^2 + 3x^3 - x^4\ dx = \frac{1}{6}[\frac{1}{2}x^2 - \frac{1}{3}3x^3 + \frac{1}{4}3x^4 - \frac{1}{5}x^5]\Big|_0^1 = \frac{1}{6}[\frac{1}{2} - \frac{1}{3}3 + \frac{1}{4}3 - \frac{1}{5}]$$
$$= \frac{1}{6}[\frac{1}{2} - 1 + \frac{3}{4} - \frac{1}{5}] = \frac{1}{6}[\frac{1}{2} - \frac{1}{4} - \frac{1}{5}] = \frac{1}{6}[\frac{1}{4} - \frac{1}{5}] = \frac{1}{6}[\frac{1}{20}] = \frac{1}{120}$$

2. Use spherical coordinates to **set up but do not evaluate** an iterated integral which will compute the volume of the 'teardrop': the region S lying inside of the surface given in spherical coordinates by $\rho = \cos(2\phi)$, $0 \leq \phi \leq \pi/4$. (See figure.)

The region we wish to integrate the function 1 over is given by

$$0 \leq \rho \leq \cos(2\phi) \text{ for}$$
$$0 \leq \phi \leq \pi/4 \text{ and } 0 \leq \theta \leq 2\pi$$



in spherical coordinates. We know that the the Jacobian determinant is $J = \rho^2 \sin(\phi)$, so the integral to compute the volume is given by

$$\int\int\int_R 1\ dV$$
$$= \int_0^{2\pi} \int_0^{\pi/4} \int_0^{\cos(2\phi)} \rho^2 \sin(\phi)\ d\rho\ d\phi\ d\theta\ .$$

[As extra credit...] We can, in fact, compute this integral:

$$\int_0^{2\pi} \int_0^{\pi/4} \int_0^{\cos(2\phi)} \rho^2 \sin(\phi)\ d\rho\ d\phi\ d\theta$$
$$= \int_0^{2\pi} \int_0^{\pi/4} \frac{1}{3}\rho^3 \sin(\phi)\Big|_0^{\cos(2\phi)} d\phi\ d\theta$$
$$= \int_0^{2\pi} \int_0^{\pi/4} \frac{1}{3}\cos^3(2\phi)\sin(\phi)\ d\phi\ d\theta$$

Using angle sum formulas, as can turn the integrand into something we can handle:

$$\cos^3(2\phi)\sin(\phi) = [\cos^2(2\phi)][\cos(2\phi)\sin(\phi)] = [\frac{1}{2}(1+\cos(4\phi))][\frac{1}{2}(\sin(3\phi)-\sin(\phi))]$$
$$= \frac{1}{4}[\sin(3\phi) - \sin(\phi) + \cos(4\phi)\sin(3\phi) - \cos(4\phi)\sin(\phi)]$$
$$= \frac{1}{4}\sin(3\phi) - \sin(\phi) + [\frac{1}{2}[(\sin(7\phi) - \sin(\phi)) - (\sin(5\phi) - \sin(3\phi))]$$
$$= \frac{1}{4}\sin(3\phi) - \frac{1}{4}\sin(\phi) + \frac{1}{8}\sin(7\phi) - \frac{1}{8}\sin(\phi) - \frac{1}{8}\sin(5\phi) + \frac{1}{8}\sin(3\phi)$$
$$= -\frac{3}{8}\sin(\phi) + \frac{3}{8}\sin(3\phi) - \frac{1}{8}\sin(5\phi) + \frac{1}{8}\sin(7\phi)$$

So $\displaystyle\int_0^{\pi/4} \cos^3(2\phi)\sin(\phi)\ d\phi = \int_0^{\pi/4} -\frac{3}{8}\sin(\phi) + \frac{3}{8}\sin(3\phi) - \frac{1}{8}\sin(5\phi) + \frac{1}{8}\sin(7\phi)\ d\phi$

$$= \frac{3}{8}\cos(\phi) - \frac{1}{8}\cos(3\phi) + \frac{1}{40}\cos(5\phi) - \frac{1}{56}\cos(7\phi)\Big|_0^{\pi/4}$$
$$= [\frac{3}{8}\cos(\pi/4) - \frac{1}{8}\cos(3\pi/4) + \frac{1}{40}\cos(5\pi/4) - \frac{1}{56}\cos(7\pi/4)] - [\frac{3}{8} - \frac{1}{8} + \frac{1}{40} - \frac{1}{56}]$$
$$= [\frac{3}{8}\frac{\sqrt{2}}{2} + \frac{1}{8}\frac{\sqrt{2}}{2} - \frac{1}{40}\frac{\sqrt{2}}{2} - \frac{1}{56}\frac{\sqrt{2}}{2}] - [\frac{3}{8} - \frac{1}{8} + \frac{1}{40} - \frac{1}{56}]$$

which is close enough, I think....

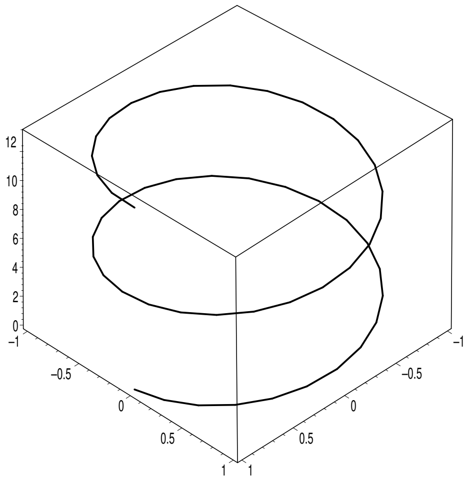
Then the volume is $\frac{1}{3}(2\pi)$ times this [for the integral $d\theta$] .

Or! Turn the integrand into products of $\cos(\phi)$ and $\sin(\phi)$, to integrate...

3. Compute the work done by the force field

$$\vec{F}(x, y, z) = (y, z, x)$$

along the curve $\gamma(t) = (\cos t, \sin t, t)$, $0 \le t \le 4\pi$.



We compute:

$\gamma'(t) = (-\sin t, \cos t, 1)$ and

$\vec{F}(\gamma(t)) = (\sin t, t, \cos t)$, so

$\vec{F}(\gamma(t)) \circ \gamma'(t) = -\sin^2 t + t\cos t + \cos t$. So

$$\int_\gamma \vec{F} \circ \, d\vec{r} = \int_0^{4\pi} -\sin^2 t + t\cos t + \cos t \, dt$$

$$= \int_0^{4\pi} -\frac{1}{2}[1 - \cos(2t)] + t\cos t + \cos t \, dt$$

$$= -\frac{1}{2}[t - \frac{1}{2}\sin(2t)] + [t\sin t + \cos t] + \sin t\Big|_0^{4\pi}$$

$$= \{-\frac{1}{2}[4\pi - \frac{1}{2}0] + [0+1] + 0\} - \{\frac{1}{2}[0 - \frac{1}{2}0] + [0+1] + 0\} = -2\pi + 1 = 1 = -2\pi$$

4. Show that the vector field

$$\vec{F}(x, y) = (x^2 + xy^2, x^2y - 3) = (F_1, F_2)$$

is a conservative vector field, and find a potential function for $\vec{F}$.

We can test for conservativicity by computing 'mixed partials':

$(F_1)_y = 0 + 2xy = 2xy = (F_2)_x$, so the vector field is conservative.

To compute a potential function, we integrate F_1, dx:

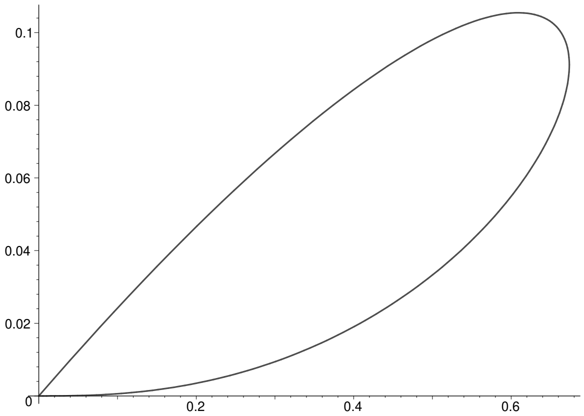
$f(x, y) = \int x^2 + xy^2 \, dx = \frac{1}{3}x^3 + \frac{1}{2}x^2y^2 + c(y)$, so

$x^2y - 3 = F_2(x, y) = f_y(x, y) = 0 + x^2y + c'(y)$, so $c'(y) = -3$, so $c(y) = -3y$.

So $f(x, y) = \frac{1}{3}x^3 + \frac{1}{2}x^2y^2 + c(y) = \frac{1}{3}x^3 + \frac{1}{2}x^2y^2 - 3y$ is a potential function for $\vec{F}$.

[As a check, we can compute that $f_x = F_1$ and $f_y = F_2$.]

5. Use Green's theorem to compute the area of the region R enclosed by the closed curve $\gamma(t) = (t + 2t^2 - 3t^3, t^3 - t^4)$, $0 \le t \le 1$. (See figure.)



We can compute the area of the region by integrating a vector field with curl equal to 1 around γ . There are many such vector fields; we'll choose $\vec{F} = (0, x)$ here.

$\gamma'(t) = (1 + 4t - 9t^2, 3t^2 - 4t^3)$, and
$\vec{F}(\gamma(t)) = (0, t + 2t^2 - 3t^3)$, so

$$\vec{F}(\gamma(t)) \circ \gamma'(t) = (t + 2t^2 - 3t^3)(3t^2 - 4t^3)$$
$$= (3t^3 + 6t^4 - 9t^5) - (4t^4 + 8t^5 - 12t^6)$$
$$= 3t^3 + 2t^4 - 17t^5 + 12t^6$$

So the area of the region R is:

$$\int_0^1 3t^3 + 2t^4 - 17t^5 + 12t^6 \, dt = \frac{3}{4}t^4 + \frac{2}{5}t^5 - \frac{17}{6}t^6 + \frac{12}{7}t^7 \Big|_0^1 = [\frac{3}{4} + \frac{2}{5} - \frac{17}{6} + \frac{12}{7}] - [0 + 0 - 0 + 0]$$

$$= \frac{15+8}{20} + \frac{72-119}{42} = \frac{23}{20} - \frac{47}{42} = \frac{483-470}{420} = \frac{13}{420} \ .$$