Math 208H, Section 1 Exam 1 Solutions

1. Find a vector of length 3 that is perpendicular to both

$$\vec{v} = \langle 1, 3, 5\rangle \text{ and } \vec{w} = \langle 2, 1, -1\rangle \ .$$

A vector perpendicular to both is given by the cross product, so we compute

$$\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 3 & 5 \\ 2 & 1 & -1 \end{vmatrix} = \begin{vmatrix} 3 & 5 \\ 1 & -1 \end{vmatrix} \vec{i} - \begin{vmatrix} 1 & 5 \\ 2 & -1 \end{vmatrix} \vec{j} + \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} \vec{k}$$

$$= \langle -3-5, -(-1-10), 1-6\rangle = \langle -8, 11, -5\rangle$$

[We can test that this is perpendicular to the two vectors by computing dot products...]

This vector has length $\sqrt{64+121+25} = \sqrt{210}$; since we want a vector of length 3, we take the appropriate scalar multiple:

$\vec{N} = \dfrac{3}{\sqrt{210}}\langle -8, 11, -5\rangle$ has length 3 and is $\perp$ tp $\vec{v}$ and $\vec{w}$. [Its negative also works...]

2. Find the **second** partial derivatives of the function $h(x, y) = x\sin(xy^2)$.

We compute: $h_x = (1)(\sin(xy^2)) + (x)(\cos(xy^2))(y^2) = \sin(xy^2) + xy^2\cos(xy^2)$

$h_y = x(\cos(xy^2))(2xy) = 2x^2y\cos(xy^2)$. Then for the second partials:

$h_{xx} - (h_x)_x = (\cos(xy^2))(y^2) + [(y^2)(\cos(xy^2)) + (xy^2)(-\sin(xy^2))(y^2)]$
$= 2y^2\cos(xy^2) - xy^4\sin(xy^2)$

$h_{xy} = h_{yx} = (h_y)_x = (4xy)(\cos(xy^2)) + (2x^2y)(-\sin(xy^2))(y^2)$
$= 4xy\cos(xy^2) - 2x^2y^3\sin(xy^2)$

$h_{yy} = (h_y)_y = (2x^2)(\cos(xy^2)) + (2x^2y)(-\sin(xy^2))(2xy)$
$= 2x^2\cos(xy^2) - 4x^3y^2\sin(xy^2)$

3. Find the equation of the plane tangent to the level **surface!**

$f(x, y, z) = x^2yz + 2y^2z - 3xy^2 = -1$ of the function f, at the point $(1, -1, 2)$.

We need the normal vector to the level surface, which is given by the gradient:

$\nabla f = (2xyz + 0 - 3y^2, x^2z + 4yz - 6xy, x^2y + 2y^2)$

Evaluating at $(1, -1, 2)$, we get $\vec{N} = (-4-3, 2-8+6, -1+2) = (-7, 0, 1)$. So the equation for the tangent plane is

$-7(x-1) + 0(y+1) + 1(z-2) = 0$, or $z = 7(x-1) + 2 = 7x - 5$.

4. Using implicit differentiation, find an equation involving the partial derviatives of f and g which implies that the graphs of the two equations $f(x, y) = c$ and $g(x, y) = d$ are perpendicular at (a, b). Use this to show that the graphs of the equations $x^2 - 2y^2 = 2$ and $x^2y = 4$ are perpendicular at their point of intersection $(2, 1)$.

By implicit differentiation we know that along $f(x,y) = c$ we have $m_1 = \dfrac{dy}{dx} = \dfrac{-f_x}{f_y}$, and along $g(x,y) = d$ we have $m_2 = \dfrac{dy}{dx} = \dfrac{-g_x}{g_y}$.

For the tangents to be perpendicular, we need these slopes to be negative reciprocals, that is, $\dfrac{-f_x}{f_y}\dfrac{-g_x}{g_y} = \dfrac{f_x g_x}{f_y g_y} = -1$, or $f_x g_x = -f_y g_y$, or $f_x g_g + f_y g_y = \nabla f \circ \nabla g = 0$. That is, in the end, we need the gradients of the two functions to be perpendicular!

In our specific case we can see that this is true: $\nabla f = \langle 2x, -4y \rangle$ and $\nabla g \langle 2xy, x^2 \rangle$, which at $(2,1)$ are $\langle 4, -4 \rangle$ and $\langle 4, 4 \rangle$, and $(4)(4) + (-4_)4) = 16 - 16 = 0$, as desired.

5. If $f(x,y) = \dfrac{x^2 y}{x+y}$, and $\gamma(t) = (x(t), y(t))$ is a parametrized curve in the domain of f with $\gamma(0) = (2,-1)$ and $\gamma'(0) = (3,5)$, then what is $\dfrac{d}{dt} f(\gamma(t))\Big|_{t=0}$?

By the chain rule, $\dfrac{df}{dt} = f_x x_t + f_y y_t$. We compute: $f_x = \dfrac{(2xy)(x+y) - (x^2 y)(1)}{(x+y)^2}$ and $f_y = \dfrac{(x^2)(x+y) - (x^2 y)(1)}{(x+y)^2}$.

At $(2,-1)$, these are $f_x = \dfrac{(-4)(1) - (-4)(1)}{(1)^2} = 0$ and $f_y = \dfrac{(4)(1) - (-4)(1)}{(1)^2} = 8$, so $\dfrac{df}{dt} = f_x x_t + f_y y_t = (0)(3) + (8)(5) = 40$.

6. For the function $g(x,y) = xy^2 - 4xy + 2x^2 + 5$, find its critical points, and, for each, determine if it is a local max, local min, or saddle point. [Hint: start with what $\dfrac{\partial g}{\partial y}$ can tell you...]

We compute: $f_x = y^2 - 4y + 4x$ and $f_y = 2xy - 4x$. Since both are always defined, we find our critical points (only) by setting both equal to 0.

$f_y = 2xy - 4x = 2x(y-2) = 0$ happens precisely when either $x = 0$ or $y = 2$. We can look at each case separately:

$x = 0$: Then $f_x = y^2 - 4y + 4x = y^2 - 4y = y(y-4) = 0$ precisely when $y = 0$ or $y = 4$. This gives us two critical points $(0,0)$ and $(0,4)$.

$y = 2$: Then $f_x = y^2 - 4y + 4x = 4 - 8 + 4x = 4x - 4 = 4(x-1) = 0$ precisely when $x = 1$. This gives us a third critical point $(1,2)$.

To determine the 'type', we need the Hessian. $f_{xx} = 4$, $f_{yy} = 2x$, and $f_{xy} = 2y - 4$, so $H = f_{xx} f_{yy} - (f_{xy})^2 = 8x - (2y-4)^2$.

At $(0,0)$, we have $H = 0 - (-4)^2 = -16 < 0$, so this is a saddle point.

At $(0,4)$, we have $H = 0 - (4)^2 = -16 < 0$, so this is also a saddle point.

At $(1,2)$, we have $H = 8 - 0^2 = 8 > 0$, and so since $f_{xx} = 4 > 0$ we have a local minimum.