Math 208H
Why Green's Theorem is true

Let $R =$ a region in the plane, and $C = \partial R =$ the boundary of R, traversed counterclockwise.

Let $F = \langle F_1, F_2 \rangle =$ a vector field on R, and let $\mathrm{curl}(F) = (F_2)_x - (F_1)_y$

Then Green's Theorem says that

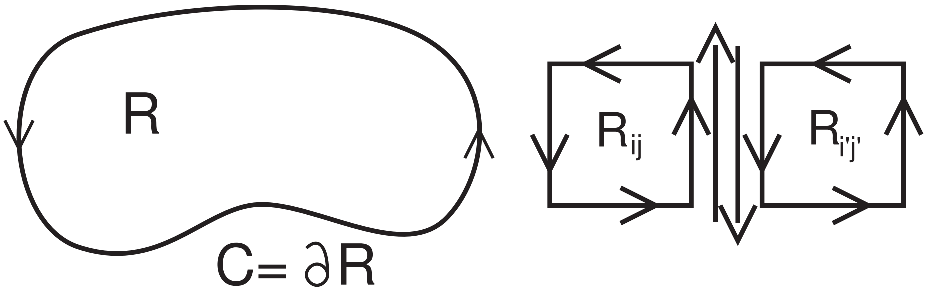
$$(*) \qquad \int\int_R \mathrm{curl}(F)\ dA = \int_C F\cdot\ dr$$

To show this, we think of R as being cut up into (or approximated by) a huge number of tiny rectangles R_{ij}.

Then (**) $\quad \int\int_R \mathrm{curl}(F)\ dA = \Sigma_{i,j} \int\int_{R_{ij}} \mathrm{curl}(F)\ dA$, since R is a "sum" of the R_{ij}'s .

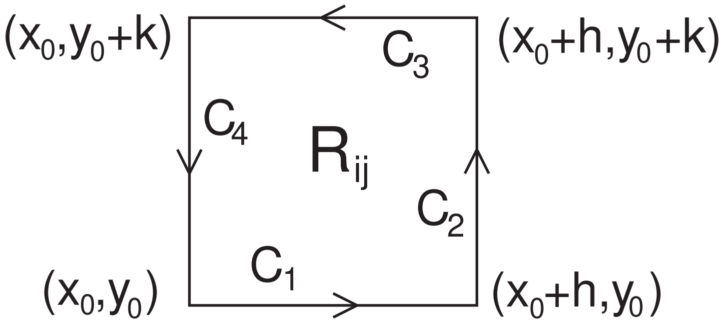
On the other hand, (***) $\quad \int_C F\cdot\ dr = \Sigma_{i,j} \int_{\partial R_{ij}} F\cdot\ dr$,

since the parts of the ∂R_{ij} that lie *inside* of R are counted twice in this sum, but are traversed in *opposite directions* when they appear. So all of the integrals over the pieces of the ∂R_{ij} cancel, *except* over the parts that traverse ∂R (which only get counted once!).



Because of these two equations (**) and (***), to verify (*) it is enough to show that
$\int\int_{R_{ij}} \mathrm{curl}(F)\ dA = \int_{\partial R_{ij}} F\cdot\ dr$

This in turn, we can do by an essentially straightforward calculation.



We can parametrize ∂R_{ij} as four pieces:

$C_1 : r_1(t) = (x_0 + t, y_0)$, $0 \le t \le h$,
$C_2 : r_2(t) = (x_0 + h, y_0 + t)$, $0 \le t \le k$,
$C_3 : r_3(t) = (x_0 + h - t, y_0 + k)$, $0 \le t \le h$,
$C_4 : r_4(t) = (x_0, y_0 + k - t)$, $0 \le t \le h$, and then

$\int_{\partial R_{ij}} F\cdot\ dr = \int_{C_1} F\cdot\ dr + \int_{C_2} F\cdot\ dr + \int_{C_3} F\cdot\ dr + \int_{C_4} F\cdot\ dr$

But, since $r_1'(t) = \langle 1, 0\rangle$, we have
$\int_{C_1} F \cdot \, dr$

$$= \int_0^h F(r_1(t)) \cdot \langle 1, 0\rangle \, dt$$
$$= \int_0^h F_1(x_0 + t, y_0) \, dt$$
$$= \int_{x_0}^{x_0+h} F_1(u, y_0) \, du$$

(using the u-substitution $\quad u = x_0 + t$),
and since $r_3'(t) = \langle -1, 0\rangle$, we have
$\int_{C_3} F \cdot \, dr$

$$= \int_0^h F(r_3(t)) \cdot \langle -1, 0\rangle \, dt$$
$$= -\int_0^h F_1(x_0 + h - t, y_0 + k) \, dt$$
$$= \int_{x_0+h}^{x_0} F_1(u, y_0 + k) \, du$$
$$= -\int_{x_0}^{x_0+h} F_1(u, y_0 + k) \, du$$

(using the u-substitution $\quad u = x_0 + h - t$) .

On the other hand,
$\int\int_{R_{ij}} \text{curl}(F) \, dA = \int\int_{R_{ij}} (F_2)_x - (F_1)_y \, dA$, and

$\int\int_{R_{ij}} -(F_1)_y \, dA$

$$= -\int_{x_0}^{x_0+h} \int_{y_0}^{y_0+k} (F_1)_y \, dy \, dx$$
$$= -\int_{x_0}^{x_0+h} (F_1(x, y)|_{y_0}^{y_0+k}) \, dx$$
$$= -\int_{x_0}^{x_0+h} F_1(x, y_0 + k) - F_1(x, y_0) \, dx$$
$$= -\int_{x_0}^{x_0+h} F_1(u, y_0 + k) \, du + \int_{x_0}^{x_0+h} F_1(u, y_0) \, du$$
$$= \int_{x_0}^{x_0+h} F_1(u, y_0) \, du - \int_{x_0}^{x_0+h} F_1(u, y_0 + k) \, du$$
$$= \int_{C_1} F \cdot \, dr + \int_{C_3} F \cdot \, dr$$

An entirely similar calculation [exercise...] shows that
$\int\int_{R_{ij}} (F_2)_x \, dA = \int_{C_2} F \cdot \, dr + \int_{C_4} F \cdot \, dr$

and so:
$\int\int_{R_{ij}} \text{curl}(F) \, dA$

$$= \int\int_{R_{ij}} (F_2)_x - (F_1)_y \, dA$$
$$= \int\int_{R_{ij}} (F_2)_x \, dA + \int\int_{R_{ij}} -(F_1)_y \, dA$$
$$= (\int_{C_2} F \cdot \, dr + \int_{C_4} F \cdot \, dr) + (\int_{C_1} F \cdot \, dr + \int_{C_3} F \cdot \, dr)$$
$$= \int_{C_1} F \cdot \, dr + \int_{C_2} F \cdot \, dr + \int_{C_3} F \cdot \, dr + \int_{C_4} F \cdot \, dr$$
$$= \int_{\partial R_{ij}} F \cdot \, dr$$

as desired!