Math 208H

A formula for the area of a polygon

We can use Green’s Theorem to find a formula for the area of a polygon $P$ in the plane with corners at the points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ (reading counterclockwise around $P$).

The idea is to use the formulas (derived from Green’s Theorem)

$$\text{Area inside } P = \int_P \langle 0, x \rangle \cdot dr = \int_P \langle -y, 0 \rangle \cdot dr$$

Each side of the polygon $P$ can be parametrized as a straight line segment $P_i$ by

$$r_i(t) = (x_i + t(x_{i+1} - x_i), y_i + t(y_{i+1} - y_i)), \quad 0 \leq t \leq 1$$

for $i = 1, \ldots, n$ (where $x_{n+1} = x_1$ is thought to have returned to the beginning of the polygon $P$, and similarly $y_{n+1} = y_1$). Then

$$\text{Area} = \int_{P_1} \langle 0, x \rangle \cdot dr = \int_{P_1} \langle 0, x \rangle \cdot dr + \cdots + \int_{P_n} \langle 0, x \rangle \cdot dr$$

(and similarly for the other integral). Then we can compute:

$$r_i'(t) = (x_{i+1} - x_i, y_{i+1} - y_i),$$

so

$$\int_{P_i} \langle 0, x \rangle \cdot dr$$

$$= \int_0^1 \langle 0, x_i + t(x_{i+1} - x_i) \rangle \cdot \langle x_{i+1} - x_i, y_{i+1} - y_i \rangle \, dt$$

$$= (y_{i+1} - y_i) \int_0^1 x_i + t(x_{i+1} - x_i) \, dt$$

$$= (y_{i+1} - y_i) (x_i t + \frac{1}{2} t^2(x_{i+1} - x_i)) \bigg|_0^1$$

$$= (y_{i+1} - y_i) (x_i + \frac{1}{2} (x_{i+1} - x_i))$$

$$= \frac{1}{2} (y_{i+1} - y_i) (x_i + x_{i+1})$$

Summing over $i$, we get our formula!

$$\text{Area} = \frac{1}{2} (y_2 - y_1)(x_1 + x_2) + \cdots + \frac{1}{2} (y_n - y_{n-1})(x_{n-1} + x_n) + \frac{1}{2} (y_1 - y_n)(x_1 + x_n)$$

This formula seems to treat the $x$’s and the $y$’s differently; one is in a sum, the other a difference. We can get a better formula if we also compute the area as

$$\text{Area} = \int_P \langle -y, 0 \rangle \cdot dr = \int_{P_1} \langle -y, 0 \rangle \cdot dr + \cdots + \int_{P_n} \langle -y, 0 \rangle \cdot dr$$

Using the exact same parametrizations for the sides, we can compute [i.e., you should compute]:

$$\int_{P_i} \langle -y, 0 \rangle \cdot dr = \frac{1}{2} (x_i - x_{i+1}) (y_i + y_{i+1})$$

so

$$\text{Area} = \frac{1}{2} (x_1 - x_2)(y_1 + y_2) + \cdots + \frac{1}{2} (x_{n-1} - x_n)(y_{n-1} + y_n) + \frac{1}{2} (x_n - x_1)(y_1 + y_n)$$
But! Since both formulas compute the same number (the area), their \textit{average} does, as well. But!

\[
\frac{1}{2} \left( \frac{1}{2} (y_{i+1} - y_i)(x_i + x_{i+1}) + \frac{1}{2} (x_i - x_{i+1})(y_i + y_{i+1}) \right)
\]

\[
= \frac{1}{4} \left( (y_{i+1} - y_i)(x_i + x_{i+1}) + (x_i - x_{i+1})(y_i + y_{i+1}) \right)
\]

\[
= \frac{1}{4} (y_{i+1} x_i - y_i x_i + y_{i+1} x_{i+1} - y_i x_{i+1} + x_i y_i - x_{i+1} y_i + x_i y_{i+1} - x_{i+1} y_{i+1})
\]

\[
= \frac{1}{4} \left( 2x_i y_{i+1} - 2x_{i+1} y_i \right)
\]

\[
= \frac{1}{2} (x_i y_{i+1} - x_{i+1} y_i)
\]

\[
= \frac{1}{2} \left| \begin{array}{cc} \frac{x_i}{y_i} & x_{i+1} \\ y_i & y_{i+1} \end{array} \right|
\]

So when we sum over these quantities, we get

\[
\text{Area} = \frac{1}{2} \left[ (x_1 y_2 - x_2 y_1) + \cdots + (x_{n-1} y_n - x_n y_{n-1}) + (x_n y_1 - x_1 y_n) \right]
\]

\[
= \frac{1}{2} \left( \left| \begin{array}{cc} x_1 & x_2 \\ y_1 & y_2 \end{array} \right| + \cdots + \left| \begin{array}{cc} x_{n-1} & x_n \\ y_{n-1} & y_n \end{array} \right| + \left| \begin{array}{cc} x_n & x_1 \\ y_n & y_1 \end{array} \right| \right)
\]

This formula “feels” better (doesn’t it?): it treats the \textit{x}-coordinates and \textit{y}-coordinates more equally. The two intermediate formulas are more lop-sided in that regard....

Hindsight is 20-20, they say; can you explain this formula differently, now that we have discovered it? A hint: each of the terms in the sum can be interpreted as the area of a triangle, with two sides equal to a certain pair of vectors....