Solutions

**Name:**

**Math 208H, Section 2**

**Final Exam**

Show all work. How you get your answer is just as important, if not more important, than the answer itself. If you think it, write it!

1. (15 pts.) Find the length of the parametrized curve
$$\vec{r}(t) = (t^6 \cos t, t^6 \sin t) \quad , \quad 0 \le t \le \pi$$

$$r'(t) = (6t^5\cos t - t^6\sin t,\ 6t^5\sin t + t^6\cos t)$$

$$\text{Length} = \int_0^{\pi}\left((6t^5\cos t - t^6\sin t)^2 + (6t^5\sin t + t^6\cos t)^2\right)^{1/2} dt$$

$$= \int_0^{\pi}\left(36t^{10}\cos^2 t - \cancel{12t^{11}\cos t\sin t} + t^{12}\sin^2 t + 36t^{10}\sin^2 t + \cancel{12t^{11}\sin t\cos t} + t^{12}\cos^2 t\right)^{1/2} dt$$

$$= \int_0^{\pi}\left(36t^{10}(\cos^2 t + \sin^2 t) + t^{12}(\sin^2 t + \cos^2 t)\right)^{1/2} dt$$

$$= \int_0^{\pi}(36t^{10} + t^{12})^{1/2} dt = \int_0^{\pi}(t^{10})^{1/2}(36+t^2)^{1/2} dt = \int_0^{\pi} t^5(t^2+36)^{1/2} dt$$

$$= \frac{1}{2}\int_0^{\pi}(t^2)^2(t^2+36)^{1/2}(2t)\,dt$$

$u = t^2+36$    $t=0,\ u=36$
$du = 2t\,dt$    $t=\pi,\ u=\pi^2+36$
$t^2 = u-36$

$$\to = \frac{1}{2}\int_{36}^{\pi^2+36}(u-36)^2 u^{1/2}\,du$$

$$= \frac{1}{2}\int_{36}^{\pi^2+36} u^{5/2} - 72u^{3/2} + 36^2u^{1/2}\,du = \frac{1}{2}\left(\frac{2}{7}u^{7/2} - 72\cdot\frac{2}{5}u^{5/2} + 36^2\cdot\frac{2}{3}u^{3/2}\right)\Big|_{36}^{\pi^2+36}$$

$$= \frac{1}{2}\left[\left(\frac{2}{7}(\pi^2+36)^{7/2} - 72\cdot\frac{2}{5}(\pi^2+36)^{5/2} + 36^2\cdot\frac{2}{3}(\pi^2+36)^{3/2}\right) - \left(\frac{2}{7}(36)^{7/2} - 72\cdot\frac{2}{5}(36)^{5/2} + 36^2\cdot\frac{2}{3}(36)^{3/2}\right)\right]$$

2. (15 pts.) Find the equation of the plane tangent to the graph of

$$z = f(x,y) = xe^y - \cos(2x+y)$$

at $(0,0,-1)$

In what direction is this plane tilting up the most?

$f_x = e^y + 2\sin(2x+y)$

$f_y = xe^y + \sin(2x+y)$

At $(0,0)$: $f_x = e^0 + 2\sin(0) = 1$

$f_y = 0\cdot e^0 + \sin(0) = 0+0 = 0$

$z-(-1) = 1\cdot(x-0) + 0\cdot(y-0) = x$

$\boxed{z = x-1}$

Most tilting? Fastest increase!

$= \nabla f(0,0) = (f_x(0,0), f_y(0,0)) = \underline{\underline{(1,0)}}$

3. (20 pts.) Find the critical points of the function

$$z = g(x, y) = x^2y^3 - 3y - 2x$$

and for each, determine if it is a local max, local min, or saddle point.

$g_x = 2xy^3 - 0 - 2 = 0$, $2xy^3 = 2$, $xy^3 = 1$

$g_y = 3x^2y^2 - 3 - 0 = 0$, $3x^2y^2 = 3$ $x^2y^2 = 1$

$x = x \cdot 1 = x(xy^3) = x^2y^3 = (x^2y^2)y = 1 \cdot y = y$

$\underline{y = x}$

$x \cdot x^3 = 1$, $x^4 = 1$, $x = 1, -1$

$(1,1)$, $(-1,-1)$ critical points

~~$f_{xx}$~~ $g_{xx} = 2y^3$, $g_{yy} = 6x^2y$, $g_{xy} = 6xy^2$

$D = g_{xx}g_{yy} - (g_{xy})^2 = (2y^3)(6x^2y) - (6xy^2)^2$

$= 12x^2y^4 - 36x^2y^4 = -24x^2y^4$

At $(1,1)$: $D = -24 \cdot 1^2 \cdot 1^4 = -24 < 0$ <u>saddle</u>

$(-1,-1)$: $D = -24(-1)^2(-1)^4 = -24 < 0$ <u>saddle</u>

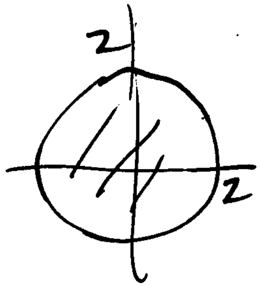
4. (15 pts.) Find the integral of the function

$$z = h(x, y) = \ln(x^2 + y^2 + 1)$$

over the region

$$R = \{(x, y) : x^2 + y^2 \leq 4\}.$$

polar! $x^2+y^2=r^2$




$$\iint_R h(x,y)\,dA = \int_0^{2\pi}\int_0^{2} \ln(r^2+1)\cdot r\,dr\,d\theta$$

$$= \int_0^{2\pi}\left(\int_0^{2} r\ln(r^2+1)\,dr\right)d\theta$$

$u=r^2+1$, $du = 2r\,dr$, $r\,dr=\frac{1}{2}du$

$r=0, u=1$; $r=2, u=5$

$$= \int_0^{2\pi}\left(\int_1^{5} \frac{1}{2}\ln u\,du\right)d\theta = \frac{1}{2}\int_0^{2\pi}(u\ln u-u)\Big|_1^5\,d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}((5\ln 5-5)-(1\ln 1-1))\,d\theta = \frac{1}{2}\int_0^{2\pi}5\ln 5-4\,d\theta$$

$$= \frac{1}{2}(5\ln 5-4)\,\theta\Big|_0^{2\pi} = \boxed{\pi(5\ln 5-4)}$$

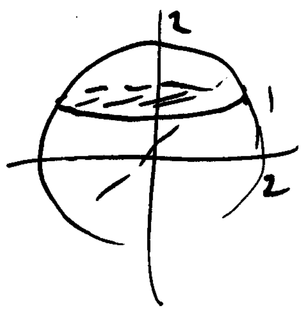
$\int \ln u\,du$ ; $v=\ln u$, $dw=du$ ; $dv=\frac{1}{u}du$, $w=u$

$$= u\ln u-\int \frac{1}{u}\cdot u\,du = u\ln u-\int du = u\ln u-u$$

5. (20 pts.) Find the integral of the function

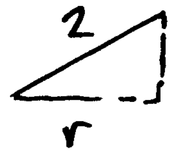
$$k(x,y,z) = z$$

over the region lying inside of the sphere of radius 2 (centered at the origin $(0,0,0)$ ) and above the plane $z = 1$ .




cylindrical coords: $x^2+y^2+z^2 = r^2+z^2 = 4$

$z = \sqrt{4-r^2}$

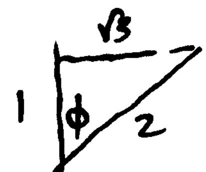



$, \ r=\sqrt{3}$.

$$\iiint_R z\,dV = \int_0^{2\pi}\int_0^{\sqrt{3}}\int_1^{\sqrt{4-r^2}} z\,dz\cdot r\,dr\,d\theta = \int_0^{2\pi}\int_0^{\sqrt{3}} r\frac{z^2}{2}\Big|_1^{\sqrt{4-r^2}} dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\sqrt{3}} \frac{r}{2}(4-r^2) - \frac{r}{2}\,dr\,d\theta = \int_0^{2\pi}\int_0^{\sqrt{3}} \frac{3r}{2} - \frac{r^3}{2}\,dr\,d\theta$$

$$= \int_0^{2\pi} \frac{3r^2}{4} - \frac{r^4}{8}\Big|_0^{\sqrt{3}} d\theta = \int_0^{2\pi}\left(\frac{9}{4}-\frac{9}{8}\right)d\theta = \int_0^{2\pi}\frac{9}{8}\,d\theta = \boxed{\frac{9\pi}{4}}$$




or $z = \rho\cos\phi = 1 \quad \rho = \frac{1}{\cos\phi} = \sec\phi \qquad \phi = \frac{\pi}{3}$

$$\iiint_R z\,dV = \int_0^{2\pi}\int_0^{\pi/3}\int_{\sec\phi}^{2} (\rho\cos\phi)(\rho^2\sin\phi)\,d\rho\,d\phi\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/3} \sin\phi\cos\phi\,\frac{\rho^4}{4}\Big|_{\sec\phi}^{2} d\phi\,d\theta = \int_0^{2\pi}\int_0^{\pi/3} 4\sin\phi\cos\phi - \frac{1}{4}\frac{\sin\phi}{\cos^3\phi}\,d\phi\,d\theta$$

$$= \int_0^{2\pi}\left(2\sin^2\phi + \frac{1}{4}\frac{(\cos\phi)^{-2}}{-2}\right)\Big|_0^{\pi/3} d\theta = (2\pi)\left(2\left(\frac{\sqrt{3}}{2}\right)^2 - \frac{1}{8}\left(\frac{1}{2}\right)^{-2}\right) - \left(0 - \frac{1}{8}(1)^{-2}\right)\Big)$$

$$= 2\pi\left(\frac{3}{2} - \frac{1}{2} + \frac{1}{8}\right) = 2\pi\left(\frac{9}{8}\right) = \boxed{\frac{9\pi}{4}}$$

6. (20 pts.) Show that the vector field $\vec{F} = \langle y^2, 2xy - 1\rangle$ is conservative, find a potential function $z = f(x, y)$ for $\vec{F}$, and use this potential function to (quickly!) find the integral of $\vec{F}$ along the path

$$\vec{r}(t) = (t\sin(2\pi t) - e^t, \ln(t^2+1) - 5t^2) \quad , \quad 0 \le t \le 1$$

$F_1 = y^2 \quad F_2 = 2xy - 1$

$(F_2)_x = 2y = (F_1)_y$ so $\text{curl}(\vec{F}) = 0$ so conservative.

$f(x,y) = \int y^2\,dx = xy^2 + g(y)$

$F_2 = 2xy - 1 = f_y = 2xy + g'(y) \quad g'(y) = 1 \quad g(y) = y$

$f(x,y) = \boxed{xy^2 - y = \text{potential function}}$

$\int_C \vec{F}\cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = f(r(1)) - f(r(0))$

$r(1) = (1\cdot\sin(2\pi) - e^1, \ln(1+1) - 5\cdot 1^2) = (-e, \ln 2 - 5)$

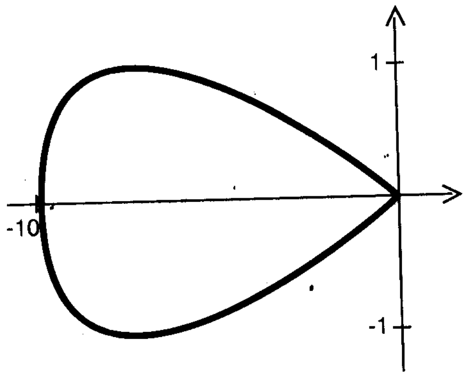
$r(0) = (0\cdot\sin(0) - e^0, \ln(0+1) - 5\cdot 0^2) = (-1, 0)$

$\int_C \vec{F}\cdot d\vec{r} = ((-e)(\ln 2 - 5)^2 - (\ln 2 - 5)) - ((-1)(0)^2 - 0)$

$= \boxed{-e(\ln 2 - 5)^2 - (\ln 2 - 5)}$

7. (15 pts.) Use Green's Theorem to find the area of the region enclosed by the curve

$$\vec{r}(t) = (t^2 - 2\pi t, \sin t) \quad , \quad 0 \le t \le 2\pi$$




$$\text{Area} = \iint_R 1\, dA = \int_C \langle -y, 0\rangle \cdot d\vec{r}$$

$$r'(t) = (2t - 2\pi, \cos t)$$

$$\text{Area} = \int_0^{2\pi} (-\sin t, 0) \cdot (2t - 2\pi, \cos t)\, dt$$

$$= \int_0^{2\pi} 2\pi \sin t - 2t \sin t\, dt$$

$$\int t \sin t\, dt \quad \begin{cases} u = t & dv = \sin t\, dt \\ du = dt & v = -\cos t \end{cases}$$

$$= -t\cos t + \int \cos t\, dt = -t\cos t + \sin t$$

$$\text{Area} = -2\pi \cos t - 2(-t\cos t + \sin t) \Big|_0^{2\pi}$$

$$= 2t\cos t - 2\pi \cos t\ 2\sin t \Big|_0^{2\pi}$$

$$= (2 \cdot 2\pi \cdot 1 - 2\pi \cdot 1 - 0) - (0 \cdot 1 - 2\pi \cdot 1 - 0)$$

$$= \boxed{4\pi}$$

Area $= \int_C \langle 0, x\rangle \cdot d\vec{r}$ gives the same answer... but takes a little longer.

8. (20 pts.) Find the flux of the vector field

$$\vec{G} = \langle x^2, xz, y\rangle$$

through that part of the graph of

$$z = f(x,y) = xy$$

lying over the rectangle

$$1 \le x \le 3 \quad , \quad 0 \le y \le 3$$

$f_x = y$

$f_y = x$

$$\iint_{\Sigma} \vec{G}\cdot\vec{N}\, dA = \int_1^3\int_0^3 \langle x^2, x(xy), y\rangle \cdot \langle -y, -x, 1\rangle\, dy\, dx$$

$$= \int_1^3\int_0^3 -x^2y - x^3y + y\, dy\, dx$$

$$= \int_1^3 -\frac{x^2y^2}{2} - \frac{x^3y^2}{2} + \frac{y^2}{2}\Big|_0^3\, dx$$

$$= \int_1^3 \left(-\frac{9}{2}x^2 - \frac{9}{2}x^3 + \frac{9}{2}\right) - (0-0-0)\, dx$$

$$= \int_1^3 \frac{9}{2} - \frac{9}{2}x^2 - \frac{9}{2}x^3\, dx = \frac{9}{2}x - \frac{9}{6}x^3 - \frac{9}{8}x^4\Big|_1^3$$

$$\boxed{= \left(\frac{9}{2}(3) - \frac{9}{6}(3)^3 - \frac{9}{8}(3)^4\right) - \left(\frac{9}{2} - \frac{9}{6} - \frac{9}{8}\right)} = -120$$