**Math 208H**

**Topics for the first exam**

**Chapter 9:** Parametric curves

The motivation: think of the graph of $y = f(x)$ as a path that we are walking along. The 'right' way to think of this is that we are visiting each point of the graph at various times $t$, e.g.,

$$x = t \ , \ y = f(x) = f(t)$$

But we need not be limited to having $x = t$ ; we can more generally describe our path as

$$x = x(t) \ , \ y = y(t)$$

This is a *parametric curve*; it describes a curve in the plane, and how we traverse it through time. The advantage is that the curve we describe need not be the graph of a function. $t$ = the parameter = the independent variable ; $x$ and $y$ = dependent variables

A circle of radius 1 centered at $(0,0)$ : $x^2 + y^2 = 1$

$$x(t) = \cos t \ , \ y(t) = \sin t \qquad 0 \leq t \leq 2\pi$$

Twice as fast around: $x(t) = \cos 2t \ , \ y(t) = \sin 2t \qquad 0 \leq t \leq \pi$

A circle of radius $r$ centered at $(a,b)$ : $(x-a)^2 + (y-b)^2 = r^2$

Think: $x - a = r\cos t \ , \ y - b = r\sin t$

$$x(t) = a + r\cos t \ , \ y(t) = b + r\sin t \qquad 0 \leq t \leq 2\pi$$

An ellipse: $(x/a)^2 + (y/b)^2 = 1$

$$x(t) = a\cos t \ , \ y(t) = b\sin t \qquad 0 \leq t \leq 2\pi$$

A line through $(a,b)$ and $(c,d)$

$$x(t) = a + t(c-a) \ , \ y(t) = b + t(d-b)$$

Finding an $(x,y)$ equation from a parametric equation: (if possible) solve for $x = x(t)$ or $y = y(t)$ as $t$ = expression in $x$ or $y$, then plug into the other equation.

Ex: $x = t^2 - 1$ , $y = t^3 + t - 1$ , then $x + 1 = t^2$ so $t = \pm\sqrt{x+1}$, so $y = (\pm\sqrt{x+1})^3 + (\pm\sqrt{x+1}) - 1$

**Calculus of curves**

Thinking of a parametric curve as a path that we are traversing, we are at each instant aware of (at least) two things: how fast we are going and what direction we are going. Each can be computed essentially as we would for a graph.

Speed = the limit of (distance)/(time interval) as the time interval shrinks to 0.

average speed = $\sqrt{(\Delta x)^2 + (\Delta y)^2}/\Delta t = \sqrt{(\Delta x/\Delta t)^2 + (\Delta y/\Delta t)^2}$

instantaneous speed = $\sqrt{(dx/dt)^2 + (dy/dt)^2} = \sqrt{(x'(t))^2 + (y'(t))^2}$

direction = slope of tangent line - limit of slopes of secant lines

secant lines: slope = $\Delta y/\Delta x = (\Delta y/\Delta t)/(\Delta x/\Delta t)$

tangent lines: slope = $(dy/dt)/(dx/dt) = y'(t)/x'(t)$

We can encode both of these in the *velocity vector* $(x'(t), y'(t))$

A parametric curve $x = x(t)$ , $y = y(t)$ , $a \leq t \leq b$ with $x(a) = x(b)$ , $y(a) = y(b)$ ends where it begins; it is a *closed curve*. Such a curve surrounds and encloses a region $R$ in the plane.

If the curve goes around the region counterclockwise, then the area of the region can be computed as

$$\text{Area} = \int_a^b x(t)y'(t)\ dt = -\int_a^b y(t)x'(t)\ dt$$

We will see why this formula is true later in this class....

**Arclength and surface area**

Just as with graphs of functions, we can compute the length of a paramentric curve and the surface area when a curve is rotated around an axis:

Length: we approximate it the same way, as a sum of lengths of line seqments that approximate the curve. Each segment has length

$$\sqrt{(\Delta x)^2 + (\Delta y/)^2} = \sqrt{(\Delta x/\Delta t)^2 + (\Delta y/\Delta t)^2}\Delta t \approx \sqrt{(x'(t))^2 + (y'(t))^2}\ dt$$

so the length of the curve is $\displaystyle\int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\ dt$

Surface area: if we spin the curve $x = x(t)$ , $y = y(t)$ , $a \leq t \leq b$ around the line $y = c$ then, just like before, we can approximate the surface by frustra of cones, each having area approximately

$2\pi|y(t) - c|\sqrt{(x'(t))^2 + (y'(t))^2}\ dt = (2\pi)(\text{radius})(\text{length})$

and so the area of the surface of revolution is

$$2\pi \int_a^b |y(t) - c|\sqrt{(x'(t))^2 + (y'(t))^2}\ dt$$

Ex: for the ellipse $x = 3\cos t$ , $y = 5\sin t$ , $0 \leq t \leq 2\pi$, spun around $y = 7$, we have

$\displaystyle\text{Area} = 2\pi \int_0^{2\pi} (7 - 3\sin t)\sqrt{9\sin^2 t + 25\cos^2 t}\ dt = 2\pi \int_0^{2\pi} (7 - 3\sin t)\sqrt{9 + 16\cos^2 t}\ dt$

**Polar coordinates**

Idea: describe points in the plane in terms of (distance,direction).

$r = (x^2 + y^2)^{1/2} = \text{distance}$ , $\theta = \arctan(y/x) = \text{angle with the positive } x\text{-axis.}$

$x = r\cos\theta$ , $y = r\sin\theta$

The same point in the plane can have many representations in polar coordinates:

$(1, 0)_{rect} = (1, 0)_{pol} = (1, 2\pi)_{pol} = (1, 16\pi)_{pol} = \ldots$

A negative distance is interpreted as a positive distance in the *opposite* direction (add $\pi$ to the angle):

$(-2, \pi/2)_{pol} = (2, \pi/2 + \pi)_{pol} = (0, -2)_{rect}$

An equation in polar coordinates can (in principal) be converted to rectangular coords, and vice versa:

E.g., $r = sin(2\theta) = 2\sin\theta\cos\theta$ can be expressed as

$r^3 = (x^2 + y^2)^{3/2} = 2(r\sin\theta)(r\cos\theta) = 2yx$, i.e., $(x^2 + y^2)^3 = 4x^2y^2$

Given an equation in polar coordinates

$r = f(\theta)$ , i.e., the curve $(f(\theta), \theta)_{pol}, \theta_1 \leq \theta \leq \theta_2$

we can compute the slope of its tangent line, by thinking in rectangular coords:

$x = f(\theta)\cos\theta, y = f(\theta)\sin\theta$ , so

$\displaystyle\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}$

Arclength: the polar curve $r = f(\theta)$ is really the (rectangular) parametrized curve

$x = f(\theta)\cos\theta, y = f(\theta)\sin\theta$, and $(x'(\theta))^2 + (y'(\theta))^2)^{1/2} = (f'(\theta))^2 + (f(\theta))^2)^{1/2}$,

so the arclength for $a \leq \theta \leq b$ is $displaystyle \int_a^b (f'(\theta))^2 + (f(\theta))^2)^{1/2}\ d\theta$

Area: if $r = f(\theta)$ , $a \leq \theta \leq b$ describes a closed curve ($f(a) = f(b) = 0$), then we can compute the area inside the curve as a sum of areas of sectors of a circle, each with area approximately

$$\pi r^2(\Delta\theta/2\pi = \frac{(f(\theta))^2}{2}\Delta\theta$$

so the area can be computed by the integral $\displaystyle\int_a^b \frac{1}{2}(f(\theta))^2\ d\theta$

**Chapter 10: Vectors**

**Vectors**

In one-variable calculus, we make a distinction between speed and velocity; velocity has a direction (left or right), while speed doesn't. Speed is the *size* of the velocity. This distinction is even more important in higher dimensions, and motivates the ntion of a *vector.*

Basically, a vector $\vec{v}$ is an arrow pointing *from* one point in the plane (or 3-space or ...) *to* another. A vector is thought of as pointing frm its *tail* to its *head.* If it points from $P$ to $Q$, we call the vector $\vec{v} = \overrightarrow{PQ}$.

A vector has both a *size* (= length = distance from $P$ to $Q$) and a *direction.* Vectors that have the same size and point in the same direction are often thought of as the same, even if they have different tails (and heads). Put differently, by picking up the vector and translating it so that its tail is at the origin (0,0), we can identify $\vec{v}$ with a point in the plane, namely its head $(x, y)$, and write $\vec{v} = \langle x, y\rangle$. If $\vec{v}$ goes from $(a, b)$ to $(c, d)$, then we would have $\vec{v}= \langle c - a, d - b\rangle$. The length of $\vec{v} = \langle a, b\rangle$ is then $||\vec{v}|| = \sqrt{a^2 + b^2}$.

In 3-space we have three special vectors, pointing in the direction of each coordinate axis (in the plane there are, analogously, two); these are called

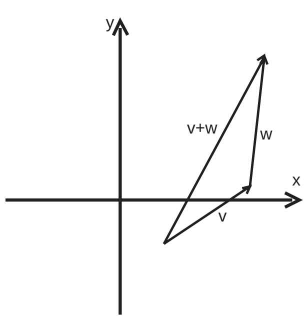
$$\vec{i} = \langle 1, 0, 0\rangle,\ \vec{j} = \langle 0, 1, 0\rangle, \text{ and } \vec{k} = \langle 0, 0, 1\rangle$$

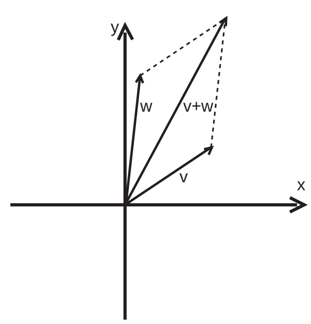
These come in especially handy when we start to add vectors. There are several different points of view to vector addition:

(1) move the vector $\vec{w}$ so that its head is on the tail of $\vec{v}$; then the vector $\vec{v} + \vec{w}$ has tail equal to the tail of $\vec{v}$ and head equal to the head of $\vec{w}$;

(2) move $\vec{v}$ and $\vec{w}$ so that their tails are both at the origin, and build the parallelogram which has sides equal to $\vec{v}$ and $\vec{w}$; then $\vec{v} + \vec{w}$ is the vector that goes from the origin to the opposite corner of the parallelogram;

(3) if $\vec{v} = \langle a, b\rangle$ and $\vec{w} = \langle c, d\rangle$, then $\vec{v} + \vec{w} = \langle a + c, b + d\rangle$




We can also subtract vectors; if they share the same tail, $\vec{v} - \vec{w}$ is the vector that points from the head of $\vec{w}$ to the head of $\vec{v}$ (so that $\vec{w} + (\vec{v} - \vec{w}) = \vec{v}$). In coordinates, we simply subtract the coordinates.

We can also *rescale* vectors = multiply them by a constant factor; $a\vec{v}$ = vector pointing in the same direction, but $a$ times as long. (We use the convention that if $a < 0$, then $a\vec{v}$ points in the *opposite* direction from $\vec{v}$.)

Using coordinates, this means that $a\langle x, y\rangle = \langle ax, ay\rangle$ . To distinguish $a$ from the coordinates or the vector, we call $a$ a *scalar.* One consequence of this formula is that $||c\vec{v}|| = |c| \cdot ||\vec{v}||$ .

All of these operations satisfy all of the usual properties you would expect:

$\vec{v} + \vec{w} = \vec{w} + \vec{v}$
$(\vec{v} + \vec{w}) + \vec{u} = \vec{v} + (\vec{w} + \vec{u})$
$a(b\vec{v}) = (ab)\vec{v}$
$a(\vec{v} + \vec{w}) = a\vec{v} + a\vec{w}$

If all that we are interested in about a vector is its *direction*, then we can choose a vector of length one pointing in the same direction:

$$\vec{u} = \frac{\vec{v}}{||\vec{v}||} = \text{unit vector pointing in the same diection as } \vec{v}\ .$$

Of course there is nothing special in all of this about vectors in the plane; all of these ideas work for vectors in 3-space. The only thing we really need to determine is the right formula for *length*: a few applications of the Pythagorean theorem leads us to

$$||\langle a, b, c\rangle|| = (a^2 + b^2 + c^2)^{1/2}$$

**Dot products**

One thing we haven't done yet is multiply vectors together. It turns out that there are two ways to reasonably do this, serving two very different sorts of purposes.

The first, the dot product, is intended to measure the extent to which two vectors $\vec{v}$ and $\vec{w}$ are pointing in the same direction. It takes a pair of vectors $\vec{v} = \langle v_1, \ldots, v_n\rangle$ and $\vec{w} = \langle w_1, \ldots, w_n\rangle$, and gives us a *scalar* $\vec{v} \bullet \vec{w} = v_1w_1 + \cdots + v_nw_n$.

Note that $\vec{v} \bullet \vec{v} = v_1^2 + \cdots + v_n^2 = ||\vec{v}||^2$. In general, $\vec{v} \bullet \vec{w} = ||\vec{v}|| \cdot ||\vec{w}|| \cdot \cos(\theta)$, where $\theta$ is the angle between the vectors $\vec{v}$ and $\vec{w}$ (when they have the same tail); this can be seen by comparing the Law of Cosines to the formula

$$||\vec{v} - \vec{w}||^2 = ||\vec{v}||^2 + ||\vec{w}||^2 - 2\vec{v} \bullet \vec{w}$$

This in turn allows us to compute this angle:

The *angle* $\Theta$ between $v$ and $w$ = the angle (between 0 and $\pi$ with $\cos(\Theta) = \langle v, w\rangle/(||v|| \cdot ||w||)$

The dot product satisfies some properties which justify calling it a product:

$\vec{v} \bullet \vec{w} = \vec{w} \bullet \vec{v}$
$(k\vec{v}) \bullet \vec{w} = k(\vec{v} \bullet \vec{w})$
$\vec{v} \bullet (\vec{w} + \vec{u}) = \vec{v} \bullet \vec{w} + \vec{v} \bullet \vec{u}$

Two vectors are orthogonal (= perpendicular) if the angle $\theta$ between them is $\pi/2$, so $\cos(\theta)$=0; this means that $\vec{v} \bullet \vec{w} = 0$. We write $\vec{v} \perp \vec{w}$.

Since $|\cos\theta| \leq 1$, we always have $|\vec{v}\bullet\vec{w}| \leq ||\vec{v}||||\vec{w}||$ . This is the *Cauchy-Schwartz inequality.* From this we can also deduce the *Triangle inequality* : $||\vec{v} + \vec{w}|| \leq ||\vec{v}|| + ||\vec{w}||$ .

Projecting one vector onto another:

The idea is to figure out how much of one vector $\vec{v}$ points *in* the direction of another vector $\vec{w}$. The dot product measures to what extent they are pointing in the same direction, so it is only natural that it plays a role.

What we wish to do is to write $\vec{v} = c\vec{w} + \vec{u}$, where $\vec{u} \perp \vec{w}$ (i.e., write $\vec{v}$ as the part pointing in the direction of $\vec{w}$ and the part $\perp \vec{w}$). By solving the equation $(\vec{v} - c\vec{w}) \bullet \vec{w} = 0$, we find that $c = (\vec{v} \bullet \vec{w})/(\vec{w} \bullet \vec{w})$.

We write $c\vec{w} = \text{proj}_{\vec{w}}\vec{v} = \dfrac{\vec{v} \bullet \vec{w}}{\vec{w} \bullet \vec{w}}\vec{w} = \dfrac{\vec{v} \bullet \vec{w}}{||\vec{w}||}\dfrac{\vec{w}}{||\vec{w}||}$ = (orthogonal) projection of $\vec{v}$ onto $\vec{w}$ .

$\vec{u} = \vec{v} - c\vec{w}$ = the part of $\vec{v}$ perpendicular to $\vec{w}$ .

**The cross product**

The dot product takes two vectors and spits out a scalar. For vectors in 3-space, there is another product, which spits out another vector. The basica idea is that given two vectros in 3-space, there is a third vector which is perpendicular to both of them. Given the two vectors

$\vec{v} = \langle a_1, a_2, a_3 \rangle$ , $\vec{w} = \langle b_1, b_2, b_3 \rangle$

we can solve the pair of equations $a_1x + a_2y + a_3z = 0$, $b_1x + b_2y + b_3z = 0$ to find that

$\langle a_2b_3 - a_3b_2, -(a_1b_3 - a_3b_1), a_1b_2 - a_2b_1 \rangle$ is a solution.

We call this vector the *cross product* $\vec{v} \times \vec{w}$ for $\vec{v}$ and $\vec{w}$ .

How do you remember this formula? Most people remember it using the notation

$$\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \vec{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \vec{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \vec{k}$$

where $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$ is the *determinant* of the $2 \times 2$ matrix

The cross product satisfies several useful equalities:

$\vec{v} \bullet (\vec{v} \times \vec{w}) = 0, \vec{w} \bullet (\vec{v} \times \vec{w}) = 0$

$\vec{v} \times \vec{w} = -\vec{w} \times \vec{v}$

$(k\vec{v}) \times \vec{w} = k(\vec{v} \times \vec{w})$

$\vec{v} \times (\vec{w} + \vec{u}) = \vec{v} \times \vec{w} + \vec{v} \times \vec{u}$

$\vec{u} \bullet (\vec{v} \times \vec{w}) = \vec{v} \bullet (\vec{w} \times \vec{u}) = \vec{w} \bullet (\vec{u} \times \vec{v})$ (the *triple product*)

$\vec{u} \times (\vec{v} \times \vec{w}) = (\vec{u} \bullet \vec{w})\vec{v} - (\vec{u} \bullet \vec{v})\vec{w}$

For our standard vectors in 3-space we have

$\vec{i} \times \vec{j} = \vec{k}$ , $\vec{j} \times \vec{k} = \vec{i}$ , $\vec{k} \times \vec{i} = \vec{j}$

Our formula for the cross product was worked out just by solving a pair of equations; any other multiple of our vector would have been perpendicular to $\vec{v}$ and $\vec{w}$, too. But in a precise sense, the formula we came up with is the right one, because the length of our vector has geometric significance:

$||\vec{v} \times \vec{w}|| = ||\vec{v}||||\vec{w}|| \sin\theta$ , where $\theta$ = angle between $\vec{v}$ and $\vec{w}$ .

But! The area of a parallelogram with sides equal to the vectors $\vec{v}$ and $\vec{w}$ is

Area = (base)$\times$(height) = $||\vec{w}|| \cdot h = ||\vec{w}|| \cdot ||\vec{v}|| \cdot \sin(\theta)$ (from trigonometry).

So: $||\vec{v} \times \vec{w}|| = ||\vec{v}|| \cdot ||\vec{w}|| \cdot \sin(\theta)$ = the area of that parallelogram!

The cross product can be used to carry out many calculations which we will find useful. For example, to compute the distance $d$ from a point $P$ to the line through the points $Q$ and $R$, we find that (setting $\vec{v} = \overrightarrow{QP}$ , $\vec{w} = \overrightarrow{QR}$) using right triangles we have

$d = ||\vec{v}|| \sin\theta = (||\vec{v}||||\vec{w}|| \sin\theta)/||\vec{w}|| = ||\vec{v} \times \vec{w}||/||\vec{w}||$

Also, to compute the volume of a parallelopiped with sides $\vec{u}, \vec{v}, \vec{w}$, we can compute

volume = (area of base)$\cdot$(height) = $||\vec{u} \times \vec{v}|| \cdot (||\vec{w}||| \cos\psi|)$

where $\psi$ = angle between $\vec{u} \times \vec{v}$ and $\vec{w}$, so

volume = $|(\vec{u} \times \vec{v}) \bullet \vec{w}|$ = absolute value of the triple product!

**Lines and planes in 3-space**

Just as with lines in the plane, we can parametrize lines in space, given a point on the line, $P$, and the direction $\vec{v}$ that the line is travelling:

$L(t) = (x(t), y(t), z(t)) = P + \vec{v}t = (x_0 + at, y_0 + bt, z_0 + ct)$

This involves a (somewhat arbitrary) parameter $t$ to describe; we can find a more *symmetric* description of the line by determining, for each coordinate, what $t$ is and setting them all equal to one another:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

To determine if and where two lines in space intersect, if we use the parametrized forms, we need to remember that the two lines might pass through that same point at different times, and so we really need to use different names for the parameters:

$P + \vec{v}t = Q + \vec{w}s$

This gives us three equations (each of the three coordinates) with two variables; it therefore usually does not have a solutions. Two lines in 3-space that do not meet are called *skew*. If two lines do meet, then we can treat them much like in the plane; we can, for example, determine the angle at which they meet by computing the angle between their direction vectors $\vec{v}, \vec{w}$ .

For planes, three points $P$, $Q$ and $R$ that do not lie on a single line will have exactly one plane through them. To describe that plane, we can think of it as all points $X$ so that $\overrightarrow{PX}$ can be expressed as a combination of $\overrightarrow{PQ}$ and $\overrightarrow{PR}$. This in turn means that $\overrightarrow{PX}$ is perpendicular to anything that is simultaneously perpendicular to both $\overrightarrow{PQ}$ and $\overrightarrow{PR}$. But the cross product is such a vector; and so we can describe the plane by insisting that

$\overrightarrow{PX} \bullet (\overrightarrow{PQ} \times \overrightarrow{PR}) = 0$

If we write $\overrightarrow{PQ} \times \overrightarrow{PR} = \langle a, b, c \rangle$ and $\overrightarrow{PX} = \langle x - x_0, y - y_0, z - z_0 \rangle$, then this equation becomes

$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$

What is really needed to describe this plane, in some sense, is the point $P = (x_0, y_0, z_0)$ and the vector $\vec{(N)} = \langle a, b, c \rangle =$ the *normal* vector to the plane. In other words, to completely describe a plane we can also use knowledge of a single point that the plane passes through, $P$, and what direction "up" is, namely the vector $\vec{(N)}$ perpendicular to the plane (i.e, the vector perpendicular to every vector lying in the plane). We can then write the equation for the plane as

$\langle x, y, z \rangle \bullet \vec{N} = P \bullet \vec{N}$

Note that if we are given the equation for the plane, we can quickly read off its normal vector; it is the coefficients of $x$, $y$, and $z$.

Intersecting planes: typically, two planes will intersect in a line (unless they are parallel, i.e., their normals are multiples of one another). We can find the parametric equation for the line by solving each equation of the plane for $x$, say, as an expression in $y$ and $z$. Setting these two expressions equal, we can express $y$, say, as a function of $z$. Plugging back into our original expression for $x$, we get $x$ as a function of $z$. So $x$, $y$, and $z$ have all been expressed in terms of a single variable, $z$, which is exactly what a parametric equation does! The direction vector for this line, it is worth pointing out, is the cross product of the normals to the two planes; this direction vector points in a direction lying in both planes, and so much be perpendicular to both normals.