Math 208

Topics for third exam

Technically, everything covered on the first two exams, *plus*

Chapter 15: Multiple Integrals

§4: Double integrals with polar coordinates

Polar coordinates describe a point in the plane by *distance* and *direction*, r and θ. We can translate from rectangular to polar coordinates by

$$(x, y) = (r\cos\theta, r\sin\theta)$$

We can use this new coordinate system to simplify some integration problems, in part because a circular disk is a *polar rectangle*, defined by $0 \leq r \leq R_0$ and $0 \leq \theta \leq 2\pi$. Similarly, circular sectors can be described as 'polar rectangles'.

But in so doing, we must interpret dA in terms of dr and $d\theta$; this is completely analogous to what we must do with u-substitution. If we have a small circular sector, made between the circles of radius r and $r + \Delta r$, and between the lines making angles θ and $\theta + \Delta\theta$, it has area approximately $r\Delta r\Delta\theta$; so

$$dA = r\ dr\ d\theta$$

and so $\displaystyle\int_R f(x, y)\ dA = \int_D f(r\cos\theta, r\sin\theta)\ r\ dr\ d\theta$, where D is how we describe the region *R in polar coordinates.*

For example, the integral of the function $f(x, y) = xy$ on the semicircle lying between the x-axis and y=$\sqrt{9 - x^2}$ can be computed as

$$\int_0^\pi \int_0^3 (r\cos\theta)(r\sin\theta)r\ dr\ d\theta$$

§5: Triple integrals with spherical and cylindrical coordinates

We can in fact easily impose *two* new coordinate systems on 3-space; each can sometimes be used to simplify an integration problem, usually by simplifying the *region* we integrate over.

With cylindrical coordinates, we simply replace (x, y, z) with (r, θ, z), i.e., use polar coordinates in the xy-plane. In the new coordinate system, $dV = (r\ dr\ d\theta)\ dz$, since that will be the volume of a small 'cylinder' of height dz lying over the small sector in the xy-plane that we use to compute dA above.

Usually, we will actually integrate in cylindrical coordinates in the order $dz\ dr\ d\theta$, since this coordinate system is most useful when the cross-sections z=constant of our region are *disks* (so the limits of integrations for z will depend only on r).

Spherical coordinates are much like polar coordinates; we describe a point (x, y, z) by distance (which we call ρ and direction, except we need to use *two* angles to completely specify the direction; first, the angle θ that $(x, y, 0)$ makes with the x-axis in the xy-plane, and then the angle ϕ that the line through our point makes with the (positive) z-axis (which we can always assume lies between 0 and π). A little trigonometry leads us to the formulas

$$x, y, z) = (\rho\cos\theta\sin\phi, \rho\sin\theta\sin\phi, \rho\cos\phi)$$

Again, the idea is that regions difficult to describe in rectangular coordinates can be far easier to describe spherically; for example, a sphere of radius R_0 can be described as the *rectangle* $0 \leq \rho \leq R_0$, $o \leq \theta \leq 2\pi$, and $0 \leq \phi \leq \pi$.

It is a bit more trouble to work out what dV is in spherical coordinates; it turns out to be

$$dv = \rho^2 \sin\phi\ d\rho\ d\theta\ d\phi$$

.

So the 'change of variables formula' for spherical coordinates reads:

$$\int_W f(x,y,z)\ dV = \int_R f(\rho\cos\theta\sin\phi, \rho\sin\theta\sin\phi, \rho\cos\phi)\ \rho^2\sin\phi\ d\rho\ d\theta\ d\phi$$

So, for example, the integral of the function $f(x,y,z) = xz$ over the top half of a sphere of radius 5 could be computed as

$$\int_0^5 \int_0^{2\pi} \in_0^{\pi/2}\ (\rho\cos\theta\sin\phi)(\rho\cos\phi)\ (\rho^2\sin\phi)\ d\rho\ d\theta\ d\phi$$

Chapter 16: Parametrized curves

§1: Parametrized curves

So far, we have talked about functions of several variables; functions which need several inputs in order to get a single output. Our next topic is *parametrized curves*; functions which have one input but several outputs. We will focus on functions of the form

$$\vec{r}(t) = (x(t), y(t)) \text{ or } \vec{r}(t) = (x(t), y(t), z(t))$$

i.e., curves in the plane or 3-space. If we think of t as time, then what $\vec{r}$ does is give us a point in the plane or 3-space at each moment of time. Thinking of $\vec{r}$ as the position of a particle, the particle sweeps out a path or curve, C, in the plane or 3-space as time passes; we think of $\vec{r}$ as *parametrizing* this curve C.

We therefore make a distinction between a *curve* (= a collection of point laid out in a string) and a *parametrized curve* (= a function which traces out a curve). A single curve can have many different parametrizations; for example,

$\vec{r_1}(t) = (\cos t, \sin t)\ ,\ 0 \leq t \leq 2\pi$
$\vec{r_1}(t) = (\cos 2t, \sin 2t)\ ,\ 0 \leq t \leq \pi$
$\vec{r_1}(t) = (\sin t, \cos t)\ ,\ 0 \leq t \leq 2\pi$
$\vec{r_1}(t) = (\cos t^2, \sin t^2)\ ,\ 0 \leq t \leq \sqrt{2\pi}$

all parametrize the (unit) circle in the plane. Their diffences with the first are that they go twice as fast, or travel in the opposite direction, or starts slowly and then moves faster and faster, respectively.

Of special interest are *lines*; they can be described as having a starting place and a direction they travel, and so can be parametrized by $\vec{r}(t) = P + t\vec{v}$, where P is the starting point and $\vec{v}$ is the direction (for example, the differencve of two points lying *along* the line).

As with ordinary functions, we can build new parametrized curves from old ones by, for example, adding constants to each coordinate (which translates the curve by those amounts), or multiplying coordinates by constants (which streches the curve in those directions).

§2: Velocity and acceleration

When we think of t as time, we can imagine ourselves as travelling along the parametrized curve $\vec{r}(t)$, and so at each point we can make sense of both *velocity* and *acceleration.* Velocity, which is the instantaneous rate of change of position, can be easily calculated from our parametrization $\vec{r}(t) = x(t), y(t), z(t)$ as $\vec{v}(t) = \vec{r}'(t) = x'(t), y'(t), z'(t)$

Similarly, acceleration can be computed as $\vec{a}(t) = \vec{r}''(t) = x''(t), y''(t), z''(t)$

On useful fact: if the length of the velocity (i.e., its *speed*), $||\vec{v}(t)||$ is constant, then $\vec{a}(t)$ is always perpendicular to $\vec{v}(t)$

And speaking of length, we can compute the *length* of a parametrized curve can be computed by integrating its speed: the length of the parametrized curve $\vec{r}(t)$, $a \leq t \leq b$, is

$$\text{Length} = \int_a^b ||\vec{v}(t)||\ dt$$

Chapter 17: Vector fields

§1: **Vector fields**

A vector field is a field of vectors, i.e., a choice of vector $F(x,y)$ (or $F(x,y,z)$) in the plane for every point in some part of the plane (the domain of F), and similarly in 3-space. We can think of F as $F(x,y) = (F_1(x,y), F_2(x,y))$; each coordinate of F is a function of several variables. We can represent a vector field pictorially by place the vector $F(x,y)$ in the plane with its *tail* at the point (x,y) . A vector field is therefore a choice of a direction (and magmitude) at each point in the plane (or 3-space...). Such objects naturally occur in many disciplines, e.g., a vector field may represent the wind velocity at each point in the plane, or the direction and magnitude of the current in a river.

One of the most important class of vector fields that we will encounter are the *gradient vector fields*. If we have an (ordinary) function $f(x,y,z)$ of several variables, then for each point (x,y,z), $\nabla(f)$ can be thought of as a *vector*, which we have in fact already taken to drawing with its tail at the point (x,y,z) (so that, for example, we can use it as a normal vector for the tangent plane to the graph of f). Many vector fields are gradient vector fields, e.g., $(y,x) = \nabla(xy)$; one of the question we will need to answer is 'How do you tell when a vector field is a gradient vector field?'. We shall see several answers to this question in the next chapter.

Chapter 18: Line Integrals

§1: **The basic idea**

We introduced vector fields $F(x,y)$ in the previous chapter in large part because these are the objects that we can most naturally integrate over a (parametrized) curve. The reason for this is that along a curve we have the notion of a velocity vector $\vec{v}$ at each point, and we can *compare* these two vectors, by taking their dot product. This tells us the extent to which F points in the direction of $\vec{v}$. Integration is all about taking averages, and so we can think if the integral of F over the curved C as measuring the *average* extent to which F points in the same direction as C.

We can set this up as we have all other integrals, as a limit of sums. Picking points $\vec{c_i}$ strung along the curve C, we can add together the dot products $F(\vec{c_i}) \bullet (\vec{c_{i+1}} - \vec{c_i})$, and then take a limit as the lengths of the vectors $\vec{c_{i+1}} - \vec{c_i}$ between consecutive points along the curve goes to 0. We denote this number by

$$\int_C F \bullet d\vec{r}$$

Such a quantity can be interpreted in several ways; we will mostly focus on the notion of *work*. If we interpret F as measuring the amount of force being applied to an object at each point (e.g., the pull due to gravity), then $\int_C F \bullet d\vec{r}$ measures the amount of work done by F as we move along C. In other words, it measures the amount that the force field F *helped* us move along C (since moving in the same direction, it helps push us along, while when moving opposite to it, it would slow us down).

In the case that F measures the current in a river or lake or ocean, and C is a *closed* curve (meaning it begins and ends at the same point), we interpret the integral of F along C as the *circulation* around C, since it measures the extent to which the current would *push* you around the curve C.

§2: **Computing using parametrized curves**

Of course, as usual, we would never want to *compute* a line integral by taking a limit! But if we use a parametrization of C, we can interpret $\int_C F \bullet d\vec{r}$ as an 'ordinary' integral. The idea is that if we use a parametrization $\vec{r}(t)$ for C then $F(\vec{c_i}) \bullet (\vec{c_{i+1}} - \vec{c_i})$ becomes

$$F(\vec{r}(t_i)) \bullet (\vec{r}(t_{i+1}) - \vec{r}(t_i))$$

But using tangent lines, we can approximate $\vec{r}(t_{i+1}) - \vec{r}(t_i)$ by $\vec{r}'(t_i)(t_{i+1} - t_i) = \vec{r}'(t_i)\Delta y$.

so we can instead compute our line integral as

$$\int_C F \bullet d\vec{r} = \int_a^b F(\vec{r}(t)) \bullet \vec{r}'(t)\ dt$$

where $\vec{r}$ parametrizes C with $a \leq t \leq b$.

Some notation that we will occasionally use: If the vector field $F = (P, Q, R)$ and $\vec{r}(t) = (x(t), y(t), z(t))$, then $d\vec{r} = (dx, dy, dz)$, so $F \bullet d\vec{r} = Pdx + Qdy + Rdz$. So we can write $\displaystyle\int_C F \bullet d\vec{r} = \int_a^b Pdx + Qdy + Rdz$

§3: **Gradient fields and path independence**

In general, the computation of a line integral can be quite cumbersome, in part because we need to evaluate the vector field F *at the point* $\vec{r}(t)$, while can yield quite complicated formulas. But there is one class of vector fields that are really quite easy to integrate: gradient vector fields. This is because we can compute:

if $F = \nabla(f)$, then $F(\vec{r}(t)) \bullet \vec{r}'(t) = \dfrac{\partial f}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dt} + \dfrac{\partial f}{\partial z}\dfrac{dz}{dt} = \dfrac{d}{dt}(f(\vec{r}(t)))$

so $\displaystyle\int_C F \bullet d\vec{r} = \int_a^b F(\vec{r}(t)) \bullet \vec{r}'(t)\ dt = \int_a^b \frac{d}{dt}(f(\vec{r}(t)))\ dt = f(\vec{r}(b)) - f(\vec{r}(a))$. We call this the *Fundamental Theorem of Line Integrals.*

We say that a vector field f is *path-independent* (or *conservative*) if the value of a line integral over a curve C depends only on what the endpoints P, Q of C are, i.e., the integral would be the same of any *other* curve running from P to Q. Our result right above can then be interpreted as saying that gradient vector fields are conservative. What is amazing is that it turns out that every conservative vector field F is the gradient vector field for some function f. We can alctually write down the function, too (by stealing an idea from the Fundamental Theorem of Calculus...), as

$f(x, y) = \displaystyle\int_C F \bullet d\vec{r}$, where C is *any* curve from (0,0) to (x, y).

§4: **Green's Theorem**

All of which is very nice, but far too theoretical for practical purposes. What we need are simple ways to tell that a vector field is conservative, and to build the function f when it is. Luckily, this is not too hard!

First, a slight reinterpretation: a vector field F is path-independent if $\int_C F \bullet d\vec{r}$=0 for every *closed* curve C.

If F is conservative, then $F = (F_1, F_2) = (f_x, f_y)$ for some function f. But then by using the equality of mixed partials for f, we can then conclude that we *must* have $(F_1)_y = (F_2)_x$. In fact, this is enough to *guarantee* that F is conservative; this is because of *Green's Theorem*: defining the *curl* of F to be $(F_2)_x - (F_1)_y$, we have

If R is a region in the plane, and C is the boundary of R, parametrized so that we travel *counterclockwise* around R, then

$$\int_C F \bullet d\vec{r} = \int_R \text{curl}(F)\ dA$$

In particular, if the curl is 0, then the integral of F along C is always 0 for every closed curve, so F is conservative.

We can actually use this result to evaluate line integrals *or* double integrals, whichever we wish. For example, we can compute the area of a region R as a line integral, by integrating the functionb 1 over R, and then using a vector field around the boundary whose curl is 1, such as $(0,y)$ or $(-x,0)$ or $(x,2y)$ or ....

This allows us to spot conservative vector fields quite easily, but doesn't tell us how to compute the function it is the gradient *of* (called its *potential function*). But in practice this is not hard; we simply write down a function f with $\frac{\partial f}{\partial x}$=$F_1$ (e.g., $f(x,y) = \int F_1(x,y)dx$). This is actually a *family* of functions, because we have the constant of integration to worry about, which we should *really* think of as a *function of* y (because we integrated a function of two variables, dx). To figure out *which* function of y, simply take $\frac{\partial}{\partial y}$ of your function, and compare with $F_2 = \frac{\partial f}{\partial y}$; just adjust the constant of integration accordingly.

Finally, there is a similar result for vector fields in dimension 3; for $F = (F_1, F_2, F_3)$, we can define curel(F) = "$\nabla \times F$" $= ((F_3)_y - (F_2)_z, -((F_3)_x - (F_1)_z), (F_2)_x - (F_1)_y)$

Then $F = \nabla f$ exactly when curl(F) = (0,0,0) ; and we can actually *construct* f using a procedure analogous to the one we came up with for vector fields with two variables.

Chapter 19: Flux Integrals

§1: The basic idea

The basic idea is that we can also integrate vector fields (in 3-space) over a *surface*. The interpretation we will use is that we are measuring the amount of fluid flowing through a surface (e.g., a cell membrane) immersed in the fluid.

We can think of a wire-frame surface sitting in a river; we would like to compute the amount of water flowing (each second, perhaps) flowing through the surface. (Or, you can think of computing the amount of rain falling on the surface of your body...)

Our input is a (velocity) vector field F, and a surface S, described in some fashion (e.g., as the graph of a function of two vairables). The idea is that a piece of surface which is tilted with respect to the vector field will not contribute much to the total. In other words, the amount flowing through the surface is related to the extent to which the (**unti**) *normal vector* for the surface is pointing in the same direction as F. We measure this with the dot product, $F \bullet \vec{n}$. This amount is also clearly proportional to the *size* of the surface; twice as much surface will give twice as much flow. This leads us to believe that what we need to add up in order to compute the flow through the surface is $F \bullet \vec{n}\ dA$ (to take into account tilt and size). So we define the *flux integral* of a vector field F over a surface S to be

$$\int_S \vec{F} \bullet d\vec{A} = \int_S (\vec{F} \bullet \vec{n})dA$$

Now at every point of the surface S, we actually have *two* choices of unit normal vector $\vec{n}$; we will see in the next section how to make a more or less 'obvious' consistent choice of normal, the *outward pointing normal*. For example, if S is a sphere of radius R, centered at (0,0,0), the outward unit normal at (x,y,z) is just $(x/R, y/R, z/R)$. If we choose F to be this *same* vector, then it is easy to see that $F \bullet \vec{n} = 1$, and so our flux integral will just compute the area of the surface S.

§2: Computing using graphs, cylindrical, and spherical coordinates

Of course, still don't want to compute flux integrals as limits of sums, either! What we need is some approaches to calculating $\vec{n}dA$. We study three cases:

Swuppose S is the graph of a function f, having domain R in the plane. What we would really like to do is to compute the flux integral as the integral of a function over R. To do this, we note that the vector $v = (-f_x, -f_y, 1)$ is normal to the graph of f; it's the normal vector we used to express the tangent plane to the graph of f. It just so happens that $v =$

$(1, 0, f_x) \times (0, 1, f_y)$, and so its length is equal to the area of the parallelogram that these two vectors span. But!, these are exactly the paralleograms we would use to approximate the graph, i.e., this length is also dA. So, $\vec{n}dA = (-f_x, -f_y, 1)$, and so

$$\int_S F \bullet \vec{n}dA = \int_R F(x, y, f(x, y)) \bullet (-f_x, -f_y, 1)\ dx\ dy\ dz$$

We can also use cylindrical and spherical coordinates, in special cases. If S is a piece of a cylinder cylinder, given by $r = r_0$, for θ and z in some range of values R, then the outward normal at r_0, θ, z is $(\cos\theta, \sin\theta, 0)$, while $dA = r_0\ d\theta\ dz$, so

$$\int_S F \bullet \vec{n}dA = \int_R F(r_0\cos\theta, r_0\sin\theta, z) \bullet (\cos\theta, \sin\theta, 0)r_0\ d\theta\ dz$$

If S is a piece of sphere, given by $\rho = \rho_0$ for θ and ϕ in some range R of values, then the outward normal is $(\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi)$ while dA is $\rho_0^2\sin\phi\ d\theta\ d\phi$, so

$$\int_S F \bullet \vec{n}dA =$$
$$\int_R F(\rho_0\cos\theta\sin\phi, \rho_0\sin\theta\sin\phi, \rho_0\cos\phi) \bullet (\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi)\ \rho_0^2\sin\phi\ d\theta\ d\phi$$