Math 208H Fall, 2000 Project

The center of mass of a triangle

The purpose of this project is to develop a general formula for the center of mass of a triangle in the plane. The center of mass of a region R in the plane is the point $(\bar{x}, \bar{y})$, where $\bar{x}$ and $\bar{y}$ are the average x- and y-values of all of the points in the region R. As we have seen these coordinates can be computed as

$$\bar{x} = \frac{\int_R x\ dA}{\int_R 1\ dA} \text{ and } \bar{y} = \frac{\int_R y\ dA}{\int_R 1\ dA}$$

What we will develop is a formula for the coordinates of the center of mass of a triangle T with vertices at the points (x_1, y_1), (x_2, y_2), and (x_3, y_3), by using a change of variables to interpret each of the integrals involved as an integral over a single 'standard' triangle. You may choose whatever standard triangle you wish; reasonable choices might be the triangles with vertices

(0,0), (1,0), and (1,1), or
(0,0), (0,1), and (1,0)

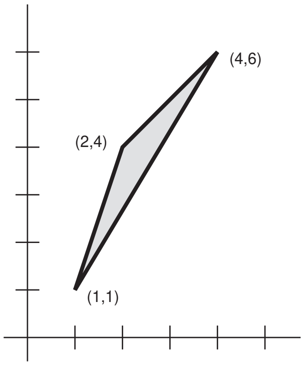
or whatever you find most convenient.

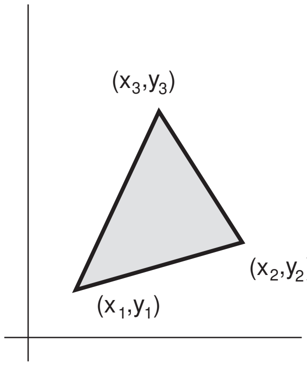
We will make use of the fact that we can map any of our standard triangles onto *any other* triangle by using the change of variables

$$x = As + Bt + C \qquad \text{and} \qquad y = Ds + Et + F$$

for suitably chosen constants A, B, C, D, E and F.

Let's start slowly. Find the center of mass of the triangle T with vertices (1,1), (4,6), and (2,4) (see figure). Set up the change of variables equations, and solve for the constants. Then using our change of variables formula, find the center of mass. Do you notice any particular relationship between the coordinates of the center of mass, and the coordinates of the vertices of the triangle?



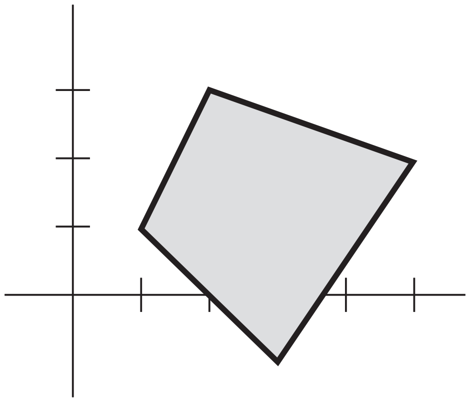


Now carry out the same calculation, but assuming we have a triangle whose vertices we only know as (x_1, y_1), etc. Solve for the constants A, etc. in terms of the coordinates of the vertices, and compute the x- and y-coordinates of the center of mass. What do the resulting formulas simplify down to?

At this point, if you had to guess, what would be your best guess for the coordinates of the center of mass for a *quadrilateral* with vertices $(x_1, y_1), \ldots, (x_4, y_4)$? To test out your hypothesis, compute the center of mass of the quadrilateral Q with vertices

(1,1), (2,3), (3,-1), (5,2)

(see figure). How do we compute it? Think of Q as being made up of *two* triangles T_1 and T_2, and compute the integral of a function F over Q as being the *sum* of the integrals of F over T_1 and T_2 . (Why should those two things be equal?). Does the resulting value match your earlier guess? If not, can you formulate a better guess based upon your extensive calculations for one quadrilateral?



When you write up your results, write it as if you were explaining the concepts and steps to someone who has taken multivariate calculus many years ago. Better yet, write as if you were writing it to *yourself*, 20 years from now. Be sure to include sufficient supporting material to make your report understandable to that future you. A page of equations, without supporting text, is certain not to be sufficient.....

You may choose to work on this project individually, or to work with some of your fellow students, in groups of up to three in size. Each group need only turn in one project, headed by all of the names of your consulting team.

Your completed report is due on your Math 208 instructor's desk by noon on Wednesday, November 29.