Math 208H

Topics from Chapter 20: Calculus of Vector Fields

§1: **The divergence of a vector field**

In terms of the coordinates $\vec{F} = (F_1, F_2, F_3)$ of a vector field, the divergence is

$$\mathrm{div}(F) = (F_1)_x + (F_2)_y + (F_3)_z$$

It can be identified with the *flux density* of the vector field $\vec{F}$ at a point P: this should be though of as the flux integral of F through a tiny box around the point P. This measures the extent to which the vector field is 'expanding', at each point.

$\mathrm{div}(F)$ = the limit as the side length goes to 0, of the flux through the sides of a box centered at P, divided by the volume of the box.

A vector field F is *divergence-free* if $\mathrm{div}(F) = 0$. For example, $F = (y, z, x)$ is divergence free, but $F = (x, y, z)$ is not; $\mathrm{div}(F) = 3$.

Some formulas that can help to calculate divergence:

$$\mathrm{div}(fF) = (\nabla f) \bullet F + f \cdot (\mathrm{div} F)$$
$$\mathrm{div}(F \times G) = (\mathrm{curl}\ F) \bullet G - F \bullet (\mathrm{curl} G) \qquad \text{in 3-space}$$
$$\mathrm{div}(\mathrm{curl}(\vec{F})) = 0 \qquad \text{in 3-space}$$

It turns out that this last result works the other way; a vector field F , defined over an entire box, which is divergence-free, is the curl of *some* other vector field G.

§2: **The Divergence Theorem**

If W is a region in 3-space, it boundary is a surface S. (S might actually consist of several pieces; this won't really effect our discussion.) We can choose normal vectors for each piece of S by insisting that $\vec{n}$ alway points *out* of W. Then we have, for any vector field F *which is defined everywhere in* W:

The Divergence Theorem: $\displaystyle\int_S \vec{F} \bullet d\vec{A} = \int_W (\mathrm{div}\ F)\ dV$

In other words, we can compute flux integrals over a surface S that forms the boundary of a region W, by computing the integral of a *different* function over W. This is especially useful when the vector field is divergence-free; for example if the region W has two surfaces for boundary and F is divergence-free, then the flux integral of F over one surface, with normals pointing out of W, is *equal* to the flux integral of F over the *other* surface, with normals pointing *into* W. Even if F is not divergence-free, we can compute the flux integral of one as the flux integral of the other *plus* the triple integral of the divergence over W.

§3: **The curl of a vector field**

We have already met the curl of a vector field $\vec{F} = (F_1, F_2, F_3$ in 3-space; in terms of coordinates:

$$\mathrm{curl}(\vec{F}) = ((F_3)_y - (F_2)_z, -((F_3)_x - (F_1)_z), (F_2)_x - (F_1)_y)$$

Its physical interpretation is as the direction where the *circulation density* of the vector field $\vec{F}$, at the point P, is the *largest.* The circulation density measures the extent to which objects caught up in a (velocity) vector field 'want' to rotate with their axis pointing in the direction of a (unit) vector $\vec{n}$, and is computed as the limit, as the side lengths go to 0, of the line integral of $\vec{F}$ around the boundary of a little square around P and *perpendicular* to $\vec{n}$, divided by the area of the square. In terms of the curl, it can be computed as

$$\text{circ}_{\vec{n}}(\vec{F}) = \text{curl}(\vec{F}) \bullet \vec{n}$$

We have already used the curl to detect conservative vector fields; this stems from the formula

$$\text{curl}(\nabla\vec{F}) = (0,0,0)$$

A vector field $\vec{F}$ is *curl-free* if curl$\vec{F} = (0,0,0)$. This means that in any *box* in which $\vec{F}$ is defined, $\vec{F}$ is a gradient vector field (although it is possible that $\vec{F}$ cannot be expressed as the gradient of a function everywhere that $\vec{F}$ is defined *at the same time*; the standard example of this is the vector field

$$\vec{F} = (\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0)$$

$\vec{F}$ is curl-free, but it is not a gradient vector field, since (as you can check) the line integral of $\vec{F}$ around the circle of radius one in the x-y plane with center (0,0,0) is 2π. Green's Theorem does not work, because $\vec{F}$ (and so its curl) is not defined on the entire disk with boundary the circle.)

§4: **Stokes' Theorem**

If S is a surface in 3-space, with a normal orientation $\vec{n}$, the boundary of S is a colection of paramatrized curves (there can easily be more than one, e.g, if S is a cylinder). We can orient each curve using a *right-hand rule*; if we stand on the curve and walk along it the chosen orientation with our heads pointing in the direction of $\vec{N}$, then the surface S dshould always be on our left. Then Stokes' Theorem says that, for any vector field $\vec{F}$ defined everywhere on S:

$$\int_C \vec{F} \bullet d\vec{r} = \int_S (\text{curl}\vec{F}) \bullet d\vec{A}$$

This allows us to compute line integrals as flux integrals, and, with a little work, flux integrals as line integrals.

For example, it says that the line integral of a curl-free vector field $\vec{F}$ around a closed curve is always 0, *so long as* the curve is the boundary of a surface contained entirely in the domain of $\vec{F}$.

We say that a vector field $\vec{F}$ is a *curl field* if $\vec{F} = \text{curl}(\vec{G})$ for some vector field $\vec{G}$. $\vec{G}$ is called a *vector potential* of $\vec{F}$. Then Stokes' Theorem says that, for any surface S in the domain of $\vec{F}$ with boundary C,

$$\int_S \vec{F} \bullet d\vec{A} = \int_S \text{curl}\vec{G} \bullet d\vec{A} = \int_C \vec{G} \bullet d\vec{r}$$

So, for example, for a curl field $\vec{F}$ and *two* surfaces S_1 and S_2 with the *same* boundary C, we have

$$\int_{S_1} \vec{F} \bullet d\vec{A} = \int_{S_2} \vec{F} \bullet d\vec{A}$$

So the flux integral of a curl field *really* depends just on the boundary of the surface, not on the surface.

We can test for whether or not $\vec{F}$ is a curl field, using the divergence, since div(curl($\vec{G}$)) = 0, so a curl field must be divergence-free. (The opposite, as we have seen, is *almost* true; it is true, for example, if the vector field is defined in a big box.)

The whole idea behind these three theorems (Green's, Divergence, and Stokes') is that the integral of one kind of function over one kind of region can be computed instead as the integral of *another* kind of function over the *boundary* of the region.

Green's: Integral of the vector field $\vec{F}$ over a closed curve in the plane equals integral of its curl of $\vec{F}$ over the region in the plane that the curve bounds.

Divergence: The flux integral of a vector field $\vec{F}$ through the boundary of a region in 3-space equals the integral of the divergence of $\vec{F}$ over the region in 3-space.

Stokes': The line integral of the vector field $\vec{F}$ over a closed curve C in 3-space equals the flux integral of the curl of $\vec{F}$ over any surface S that has C as its boundary.

Note that Green's Theorem is really just a special case of Stokes' (where the curve C lies in the plane, and the third coordinate of $\vec{F}$ just happens to be 0). All of these, like the Fundamental Theorem of Line Integrals, are really a kind of Fundamental Theorem of Calculus, where we are computing a kind of integral by instead computing something else across the boundary of the region we are interested in. We could keep doing this, finding a relation between integrals over regions in 4-space (or higher!) in terms of integrals over thier 'boundary', but we won't do that....