Math 189H Joy of Numbers Activity Log

Thursday, September 22, 2011

Bjarne Strustrup: (lecture at Temple U, 1997) *"If you think it's simple, then you have misunderstood the problem."*

Frank Capra: "Film is one of the three universal languages, the other two: mathematics and music."

1/24 of an ounce is known as a 'scruple'.

Today we continued to talk about checking your arithmetic for accuracy (without re-doing the calulation). If someone told us that $111 \cdot 123 = 13753$, or test for the correct last digit would not be able to tell us this was wrong. [The correct answer is 13653.] But a test called 'casting out nines' would succeed in noting the error. The basic idea is similar to what we did for 10; If we write

$$111 = 9n + r \text{ and } 123 = 9m + t$$

(what n and m will turn out to be not important, but what r and t we will be!), then $111 \cdot 123 = (9n+r)(9m+t) = 9(9nm+nt+rm)+rt$, and if we further write $rt = 9Q+U$, then $111 \cdot 123 = 9(9nm + nt + rm + Q) + U$. In words, the remainder that $111 \cdot 123$ will leave when divided by 9 can be computed by first finding the remainders that 111 and 123 leave (namely, r and t), multiplying <u>them</u> together (to get rt), and then finding the remainder that rt leaves when you divide by 9. In our specific case, $111 = 9 \cdot 12 + 3$ and $123 = 9 \cdot 13 + 6$, so since $3 \cdot 6 = 18 = 9 \cdot 2 + 0$, $111 \cdot 123$ leaves remainder 0 when you divide by 9. But $13753 = 9 \cdot 1528 + 1$, which leaves remainder 1 when you divide by 9, so $111 \cdot 123$ <u>cannot</u> be equal to 13753.

But that didn't seem like less work than doing the multiplication over again... The point is to find a better way to determine the remainder that a number N will leave when we divide by 9; and in particular, to figure out what the 'r' in $N = 9n + r$ is (with $0 \leq r \leq 8$) <u>without</u> determining n (!). The idea, basically, is that we can subtract multiples of 9 from N without affecting the remainder r (this would only affect the n, which we no longer 'care' about), and we can do that any way we want. To figure out how to do this efficiently, we looked at powers of 10:

$10^0 = 1 = 9 \cdot 0 + 1$
$10^1 = 10 = 9 \cdot 1 + 1$
$10^2 = 100 = 9 \cdot 11 + 1$
$10^3 = 9 \cdot 111 + 1$
$10^4 = 9 \cdot 1111 + 1$
$10^5 = 9 \cdot 11111 + 1$

After enough of these one gets suspicious: in fact, 10^k leaves remainder 1 on division by 9 for every possible positive integer k.

To streamline our later discussions on things like this, we will adopt a common terminology and notation to say and write

'n is congruent to m modulo a' (written $n \underset{a}{\equiv} m$, or $n \equiv m (\text{mod } a)$) to mean that n and m leave the same remainder when you divide by a, which is the same as $a|n-m$. [This last is of course just as short to write as $n \underset{a}{\equiv} m$, but this latter notation allows us to write things like $n \underset{a}{\equiv} m \underset{a}{\equiv} r \underset{a}{\equiv} s$, and, as we will later see, treat '$\underset{a}{\equiv}$' like equality in many other ways!]

In this notation, what we have found is that $10^k \underset{9}{\equiv} 1$ for every choice of k. What this allows us to do is to say that we can replace 10^k with 1 in any calculation where our only concern is what the remainder will be when we divide by 9:
$247 = 2 \cdot 100 + 4 \cdot 10 + 1 = 2 \cdot 10^2 + 4 \cdot 10^1 + 1$ will have the same remainder on division by 9 as $2 + 4 + 7 = 15$, since we have simply subtracted $2 \cdot (9 \cdot 11) + 4 \cdot (9 \cdot 1)$, which will not change the remainder. Carried to its logical conclusion, any number is congruent, mod 9, to the sum of its digits: each digit contributes $a \cdot 10^k$ to the number (i.e., it is a sum of such things), and each of those contribute a to the remainder when we divide by 9. This concept is really just a generalization of the rule you probably all remember: a number is divisible by 9 (i.e., leaves remainder 0) precisely when the sum of its digits is a multiple of 9 (i.e., leave the same remainder, 0). So the final version of our 'casting out nines' check is: start with two numbers, and find their remainders on division by 9 by finding the remainders of the sum of each of their digits. [Note that this can be repeated! If the sum of the digits has more than one digit, find the remainder by summing the digits of the sum! And so on, and so on...] Then multiply the remainders together, and find the reaminder of the product by the same summing approach. This remainder must agree with the remainder of our declared product, on division by 9, otherwise the computation was incorrect.

But things didn't stop there! We can use the same idea to find the remainder of any number on division by any other number, at least in principle. And this can be used to develop a test for divisibility by any number p we might like: we are checking to see if the remainder on division by p is 0. The idea, again, is to come up with an effecient way to subtract multiples of p from any particular number N, which will not change the remainder. We tried this out with $p = 7$, to develop a test for divisibility 7.

Taking our cue from 9, we looked at the remainders of the powers of 10:

$1 = 10^0 == 7 \cdot 0 + 1$
$10 = 10^1 = 7 \cdot 1 + 3$
$100 = 10^2 = 7 \cdot 14 + 2$
$1000 = 10^3 = 7 \cdot 142 + 6$
$10000 = 10^4 = 7 \cdot 1428 + 4$
$100000 = 10^5 = 7 \cdot 14285 + 5$
$1000000 = 10^6 = 7 \cdot 142857 + 1$
$10000000 = 10^7 = 7 \cdot 14285671 + 3$

Several of you noticed (before your instructor did!) that our quotients are basically longer and longer pieces of the decimal expansion of $1/7$, which makes sense, since we are looking at the integer parts of the fractions $10^k/7$, which is just $1/7$ with the decimal shifted further and further to the right. But for our purposes now it is the remainders that are of interest: the above numbers tell us, for example, that 23751 is congruent, mod 7, to $1 \cdot 1 + 5 \cdot 3 + 7 \cdot 2 + 3 \cdot 6 + 2 \cdot 4 = 56 = 7 \cdot 8 + 0$, so 23751 is, in fact, a multiple of 7. Had we kept going with our list, we would find that the remainders repeat, in fact, they cycle around:

$1, 3, 2, 6, 4, 5, 1, 3, 2, 6, 4, 5, 1, 3, 2, 6, 4, 5, \ldots$

By knowing the first 6 values (in order) and the fact that they cycle, we can create a fairly quick test for divisibilty by 7: line up the digits of our number N in reverse order, from ones digit to the highest power, then multiply them in turn by the sequence above. Then add the results of these mutiplications together to get the number r. The remainder of N on division by 7 is the same as the remainder of r (which, if it has more digits than you like, you can determine by repeating this process!). In particular N is a multiple of 7 precisely when r is.

One can in practice speed this up by remembering that, as we are building the ultimate sum out of the digits of N, we can throw away multiples of 7 at any time. So if one of the digits of N is 8, we can use 1 instead (effectively, subtracting $(8-1) \cdot 10^k = 7 \cdot 10^k$ from our number). And if, say a digit 5 is being multiplied by 5 (yielding 25), we can instead record $25 - 3 \cdot 7 = 4$, since this will not change the remainder, either.

Next time, we will take a closer look at the kind of calculations we are carrying out with all of this, to see if there are general rules, or an underlying pattern, that we can take advantage of!

To stimulate discussion for Tuesday, we finished with the following questions:

How would you devise a divisibility check for 11 along the lines of the ones we just built for 9 and 7? How about for 13?