Math 1710

Topics for third exam

Chapter 3: Applications of Derivatives

§7: **Linear approximation and differentials**

Idea: The tangent line to a graph of a function makes a good approximation to the function, near the point of tangency.

Tangent line to $y = f(x)$ at $(x_0, f(x_0)$: $L(x) = f(x_0) + f'(x_0)(x - x_0)$

$f(x) \approx L(x)$ for x near x_0

Ex.: $\sqrt{27} \approx 5 + \dfrac{1}{2 \cdot 5}(27 - 25)$, using $f(x) = \sqrt{x}$

$(1 + x)^k \approx 1 + kx$, using x_0=0

$\Delta f = f(x_0 + \Delta x) - f(x_0)$, then $f(x_0 + \Delta x) \approx L(x_0 + \Delta x)$ translates to

$\Delta f \approx f'(x_0) \cdot \Delta x$

differential notation: $df = f'(x_0)dx$

So $\Delta f \approx df$, when $\delta x = dx$ is small

In fact, $\Delta f - df =$ (diffrnce quot $-f'(x_0))\Delta x =$ (small)$\cdot$(small) = really small, goes like $(\Delta x)^2$

§8: **Newton's method**

A really fast way to approximate roots of a function.

Idea: tangent line to the graph of a function "points towards" a root of the function

Roots of (tangent) lines are easy to find!

$L(x) = f(x_0) + f'(x_0)(x - x_0)$; root is $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$

Now use x_1 as starting point for new tangent line; keep repeating!

$x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

Basic fact: if x_n approximates a root to k decimal places, then x_{n+1} tends to approximate it to $2k$ decimal places!

BUT:

Newton's method might find the "wrong" root: Int Value Thm might find one, but N.M. finds a different one!

Newton's method might crash: if $f'(x_n) = 0$, then we can't find x_{n+1} (horizontal lines don't have roots!)

Newton's method might wander off to infinity, if f has a horizontal asymptote; an initial guess too far out the line will generate numbers even farther out.

Newton's method can't find what doesn't exist! If f has no roots, Newton's method will try to "find" the function's closest approach to the x-axis; but everytime it gets close, a nearly horizontal tangent line sends it zooming off again!

Chapter 4: Integration

§1: **Antiderivatives**

Integral calculus is all about finding areas of things, e.g. the area between the graph of a function f and the x-axis. This will, in the end, involve finding a function F whose *derivative* is f.

F is an *antiderivative* (or (indefinite) *integral*) of f if $F'(x) = f(x)$.

Notation: $F(x) = \int f(x)\ \mathrm{d}x$; it <u>means</u> $F'(x)=f(x)$

"the integral of f of x dee x"

Basic list:

$\int x^n\ \mathrm{d}x = \dfrac{x^{n+1}}{n+1} + C$ (provided $n \neq -1$)

$\int \sin(kx)\ \mathrm{d}x = \dfrac{-\cos(kx)}{k} + C$

$\int \cos(kx)\ \mathrm{d}x = \dfrac{\sin(kx)}{k} + C$

$\int \sec^2 x\ \mathrm{d}x = \tan x + C$

$\int \csc^2 x\ \mathrm{d}x = -\cot x + C$

$\int \sec x \tan x\ \mathrm{d}x = \sec x + C$

$\int \csc x \cot x\ \mathrm{d}x = -\csc x + C$

Most differentiation rules can be turned into integration rules (although some are harder than others; some will even wait until Calc II !)

Basic integration rules: sum and constant multiple rules are easy to reverse

k=constant

$\int k \cdot f(x)\ \mathrm{d}x = k \int f(x)\ \mathrm{d}x$

$\int (f(x) \pm g(x)\ \mathrm{d}x = \int f(x)\ \mathrm{d}x \pm \int g(x)\ \mathrm{d}x$

§3: Integration by substiution

The idea: reverse the chain rule!

if $g(x) = u$, then $\dfrac{d}{dx} f(g(x)) = \dfrac{d}{dx} f(u) = f'(u)\ \dfrac{du}{dx}$

so $\int f'(u)\ \dfrac{du}{dx}\ dx = \int f'(u)\ du = f(u) + c$

$\int f(g(x))g'(x)\ dx$; set $u = g(x)$

then $du = g'(x)\ dx$, so $\int f(g(x))g'(x)\ dx = int f(u)\ du$, <u>where</u> $u = g(x)$

Example: $\int x(x+2-3)^4\ dx$; set $u = x^2 - 3$, so du=$2x\ dx$. Then

$$\int x(x+2-3)^4\ dx = \frac{1}{2}\int (x+2-3)^4 2x\ dx = \frac{1}{2}\int u^4\ du\ |_{u=x^2-3} =$$

$$\frac{1}{2}\frac{u^5}{5} + c\ |_{u=x^2-3} = \frac{(x^2-3)^5}{10} + c$$

The three most important points:

1. Make sure that you calculate (and then set aside) your du before doing step 2!
2. Make sure everything gets changed from x's to u's
3. <u>**Don't**</u> push x's through the integral sign! They're <u>not</u> constants!

§4: Estimating things with sums

Idea: alot of things can estimated by adding up alot of tiny pieces.

Sigma notation: $\displaystyle\sum_{i=1}^{n} a_i = a_1 + \cdots a_n$; just add the numbers up

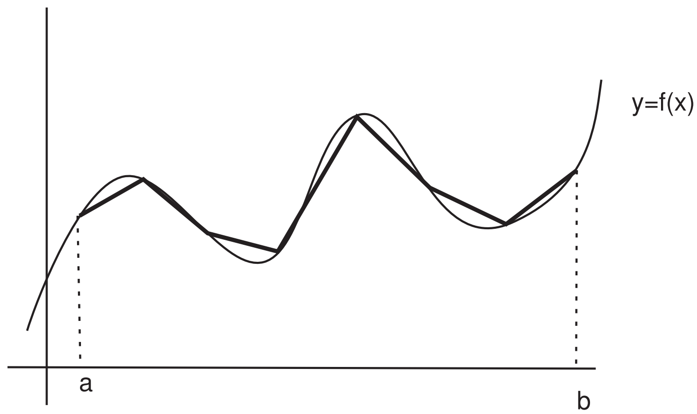
formal properties:

$$\sum_{i=1}^{n} k a_i = k \sum_{i=1}^{n} a_i$$

$$\sum_{i=1}^{n}(a_i \pm b_i) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i$$

Some things worth adding up:

length of a curve: approximate curve by a collection of straight line segments



length of curve $\approx \sum$(length of line segments)

distance travelled = (average velocity)(time of travel)

over short periods of time, avg. vel. $\approx$ instantaneous vel.

so distance travelled $\approx \sum$(inst. vel.)(short time intervals)

E.g., $s(t)$=position, $v(t)$=velocity, use velocity 4 times per second

$$\text{dist. travelled} = s(10) - s(5) \approx \sum_{i=1}^{20} v(5 + \frac{i}{4})(\frac{1}{4})$$

average value of a function

average of n numbers: add the numbers, divide by n

for a function, add up lots of values of f, divide by number of values

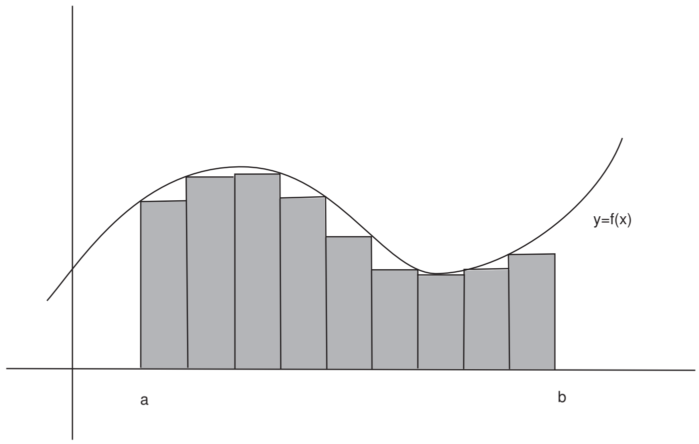
avg. value of $f \approx \frac{1}{n}\sum_{i=1}^{n} f(c_i)$

§5: **Definite integrals**

The most important thing to approximate by sums: area under a curve.

Idea: approximate region b/w curve and x-axis by things whose areas we can easily calculate:

rectangles!



Area between graph and x-axis $\approx \sum$ (areas of the rectangles) $=\sum_{i=1}^{n} f(c_i)\Delta x_i$

We define the area to be the limit of these sums as the number of rectangles goes to ∞ (i.e., the width of the rectangles goes to 0), and call this the *definite integral* of f from a to b:

$$\int_a^b f(x)\ dx \ = \ \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x_i$$

When do such limits exist?

Theorem If f is continuous on the interval $[a,b]$, then $\int_a^b f(x)\ dx$ exists.

(i.e., the area under the graph is approximated by rectangles.)

§6: **Properties of definite integrals**

Fisrt note: the sum used to define a definite integral does need to have $f(x) \geq 0$; the limit still makes sense. When f is bigger than 0, we interpret the integral as area under the graph.

Basic properties of definite integrals:

$$\int_a^a f(x)\ dx = 0$$

$$\int_b^a f(x)\ dx = -\int_a^b f(x)\ dx$$

$$\int_a^b kf(x)\ dx = k\int_a^b f(x)\ dx$$

$$\int_a^b f(x) \pm g(x)\ dx = \int_a^b f(x)\ dx \pm \int_a^b g(x)\ dx$$

$$\int_a^b f(x)\ dx + \int_b^c f(x)\ dx = \int_a^c f(x)\ dx$$

If $m \leq f(x) \leq M$ for all x in $[a,b]$, then

$$m(b-a) \leq \int_a^b f(x)\ dx \leq M(b-a)$$

More generally, if $f(x) \leq g(x)$ for all x in $[a,b]$, then

$$\int_a^b f(x)\ dx \leq \int_a^b g(x)\ dx$$

Average value of f : formalize our old idea!

$\text{avg}(f) = \dfrac{1}{b-a} \displaystyle\int_a^b f(x)\, dx$

Mean Value Theorem for integrals: If f is continuous in $[a,b]$, then there is a c in $[a,b]$ so that $f(c) = \dfrac{1}{b-a} \displaystyle\int_a^b f(x)\, dx$

§7: The fundamental theorem of calculus

Formally, $\displaystyle\int_a^b f(x)\, dx$ <u>depends</u> on a and b. Make this explicit:

$\displaystyle\int_a^x f(t)\, dt = F(x)$ is a function of x.

$F(x) =$ the area under the graph of f, from a to x.

Fund. Thm. of Calc (# 1): If f is continuous, then $F'(x) = f(x)$

(F is an antiderivative of f !)

Since any two antiderivatives differ by a constant, and $F(b) = \displaystyle\int_a^b f(t)\, dt$, we get

Fund. Thm. of Calc (# 2): If f is continuous, and F is an antiderivative of f, then

$$\int_a^b f(x)\, dx = F(b) - F(a) = F(x)\ \Big|_a^b$$

Ex: $\displaystyle\int_0^\pi \sin x\, dx = (-\cos\pi) - (-\cos 0) = 2$

Building antiderivatives:

$F(x) = \displaystyle\int_a^x \sqrt{\sin t}\, dt$ is an antiderivative of $f(x) = \sqrt{\sin x}$

$G(x) = \displaystyle\int_{x^2}^{x^3} \sqrt{1+t^2}\, dt = F(x^3) - F(x^2)$, where

$F'(x) = \sqrt{1+x^2}$, so $G'(x) = F'(x^3)(3x^2) - F'(x^2)(2x)$...

§8: substitution and definite integrals

We can use u-substitution directly with a definite integral, provided we remember that

$\displaystyle\int_a^b f(x)\, dx$ <u>really</u> means $\displaystyle\int_{x=a}^{x=b} f(x)\, dx$

and we remember to change <u>all</u> of the x's to u's!

Ex: $\displaystyle\int_1^2 x(1+x^2)^6\, dx$; set $u = 1+x^2$, $du = 2x\, dx$. when $x = 1$, $u = 2$; when $x = 2$, $u = 5$; so

$$\int_1^2 x(1+x^2)^6\, dx = \frac{1}{2}\int_2^5 u^6\, du = ...$$