Math 1710

Topics for second exam

Chapter 2: Derivatives

§7: **Related Rates**

Idea: If two (or more) quantities are related (a change in one value means a change in others), then their rates of change are related, too.

$xyz = 3$; pretend each is a function of t, and differentiate (implicitly).

General procedure:

Draw a picture, describing the situation; label things with variables.

Which variables, rates of change do you know, or want to know?

Find an equation relating the variables whose *rates of change* you know or want to know.

Differentiate!

Plug in the values that you know.

Chapter 3: Applications of Derivatives

§1: **Extreme Values**

c is an (absolute) maximum for a function $f(x)$ if $f(c) \geq f(x)$ for every other x

d is an (absolute) minimum for a function $f(x)$ if $f(d) \leq f(x)$ for every other x

max or min = extremum

Extreme Value Theorem: If f is a continuous function defined on a closed interval $[a, b]$, then f actually *has* a max and a min.

Goal: figure out where they *are*!

c is a relative max (or min) if $f(c)$ is $\geq f(x)$ (or $\leq f(x)$) for every x *near* c. Rel max or min = rel extremum.

An absolute extremum is either a rel extremum or an endpoint of the interval.

c is a critical point if $f'(c) = 0$ or does not exist.

A rel extremum is a critical point.

So absolute extrema occur either at critical points *or* at the endpoints.

So to find the abs max or min of a function f on an interval $[a, b]$:

(1) Take derivative, find the critical points.

(2) Evaluate f at each critical point and endpoint.

(3) Biggest value is maximum value, smallest is minimum value.

§2: **The Mean Value Theorem**

You can (almost) recreate a function by knowing its derivative

Mean Value Theorem: if f is containous on $[a, b]$ and differentiable on (a, b), then there is at least one c in (a, b) so that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Consequences:

Rolle's Theorem: $f(a) = f(b) = 0$; between two roots there is a critical point.

So: If a function has no critical points, it has at *most* one root!

A function with $f'(x)=0$ is constant.

Functions with the same derivative (on an interval) differ by a constant.

f is *increasing* on an interval if $x > y$ implies $f(x) > f(y)$
f is *decreasing* on an interval if $x > y$ implies $f(x) < f(y)$
If $f'(x) > 0$ on an interval, then f is increasing
If $f'(x) < 0$ on an interval, then f is decreasing

§3: **The First Derivative Test**

Local max's / min's occur at critical points; how do you tell them apart?

Near a local max, f is increasing, then decreasing; $f'(x) > 0$ to the left of the critical point, and $f'(x) < 0$ to the right.

Near a local min, the opposite is true; $f'(x) < 0$ to the left of the critical point, and $f'(x) > 0$ to the right.

If the derivative does *not* change sign as you cross a critical point, then the critical point is not a rel extremum.

Basic use: plot where a function is increasing/decreasing: plot critical points; in between them, sign of derivative does not change.

§4: **Graphing**

when we look at a graph, we see where function is increasing/decreasing. We also see:

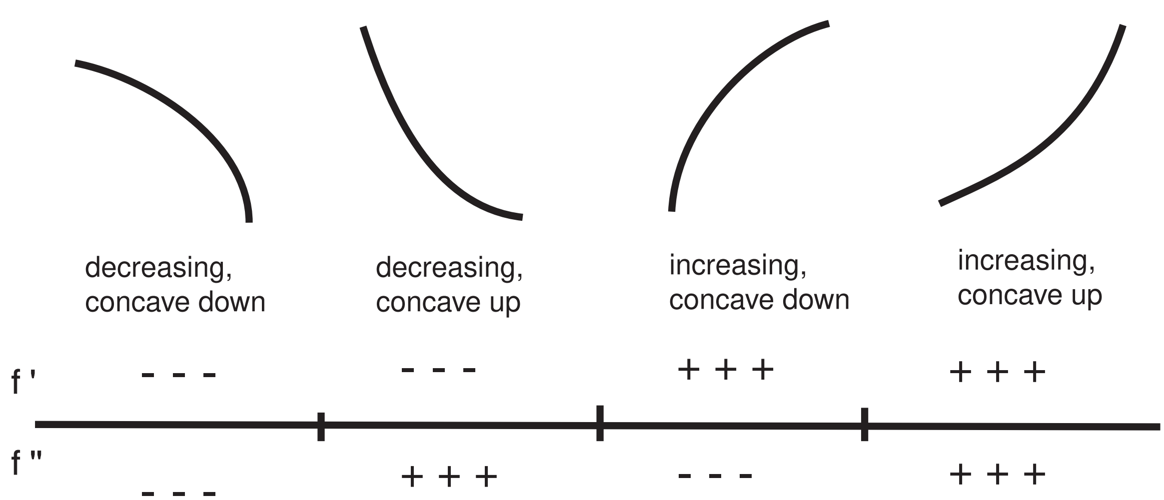
f is concave up on an interval if $f''(x) > 0$ on the interval
Means: f' is increasing; f is *bending* up.
f is concave down on an interval if $f''(x) < 0$ on the interval
Means: f' is decreasing; f is *bending* down.
A point where the concavity changes is called a point of inflection

Graphing:

Find where $f'(x)$ and $f''(x)$ are 0 or DNE

Plot on the same line.

In between points, derivative and second derivative don't change sign, so graph looks like one of:



Then string together the pieces!

Second derivative test: If c is a critical point and

$f''(c) > 0$, then c is a rel min (smiling!)

$f''(c) < 0$, then c is a rel max (frowning!)

§5: **Limits at infinity, asymptotes**

Last bit of information for a graph: what happens at the ends?

$\lim\limits_{x\to\infty} f(x) = L$ means $f(x)$ is close of L when x is really large.

$\lim\limits_{x\to-\infty} f(x) = M$ means $f(x)$ is close of M when x is really large and *negative*.

Basic fact: $\lim\limits_{x\to\infty} \frac{1}{x} = \lim\limits_{x\to-\infty} \frac{1}{x} = 0$

More complicated functions: divide by the highest power of x in the denomenator.

$f(x), g(x)$ polynomials, degree of $f = n$, degree of $g = m$

$$\lim_{x\to\pm\infty} \frac{f(x)}{g(x)} = 0 \text{ if } n < m$$

$$\lim_{x\to\pm\infty} \frac{f(x)}{g(x)} =$$

(coeff of highest power in f)/(coeff of highest power in g) if $n = m$

$$\lim_{x\to\pm\infty} \frac{f(x)}{g(x)} = \pm\infty \text{ if } n > m$$

The line $y = a$ is a horizontal asymptote for a function f if $\lim\limits_{x\to\infty} f(x)$ or $\lim\limits_{x\to-\infty} f(x)$ is equal to a.

I.e., the graph of f gets really close to $y = a$ as $x \to \infty$ or $a \to -\infty$

The line $x = b$ is a vertical asymptote for f if $f \to \pm\infty$ as $x \to b$ from the right or left.

If numerator and denomenator of a rational function have no common roots, then vertical asymptotes = roots of denom.

$f \to \infty$ or $f \to -\infty$: can use f incr or decr on either side of b to decide (so long as you already know it is blowing up!)

§6: **Optimization**

This is really just finding the max or min of a function on an interval, with the added complication that you need to figure out *which* function, and *which* interval! Solution strategy is similar to related rates:

Draw a picture; label things.

What do you need to maximize/minimize? Write down a formula for the quantity.

Use other information to eliminate variables, so your quantity depends on only one variable.

Determine the largest/smallest that the variable can reasonably be (i.e., find your interval)

Turn on the max/min machine!