Math 1710

Topics for first exam

Chapter 1: Limits and Continuity

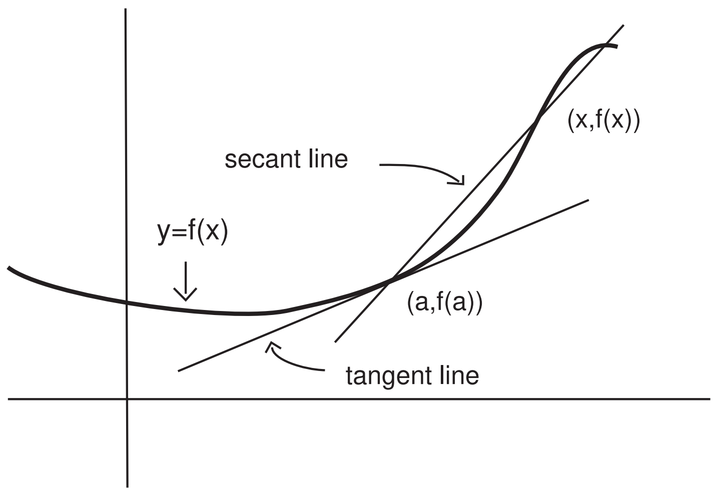
§1: **Rates of change and limits**

Calculus = Precalculus + (limits)

Limit of a function f at a point x_0 = the value the function 'should' take at the point
= the value that the points 'near' x_0 tell you f should have at x_0

$\lim_{x\to x_0} f(x) = L$ means $f(x)$ is close to L when x is close to (but not equal to) x_0

Idea: slopes of tangent lines



The closer x is to a, the better the slope of the secant line will approximate the slope of the tangent line.

The slope of the tangent line = limit of slopes of the secant lines (through (a,f(a)))

$\lim_{x\to x_0} f(x) = L$ does not care what $f(x_0)$ is; it ignores it

$\lim_{x\to x_0} f(x)$ need not exist! (function can't make up it's mind?)

§2: **Rules for finding limits**

If two functions $f(x)$ and $g(x)$ agree (are equal) for every x near a
(but maybe not at a), then $\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$

Ex.: $\lim_{x\to 2} \dfrac{x^2-3x+2}{x^2-4} = \lim_{x\to 2} \dfrac{x-1}{x+2}$

If $f(x)\to L$ and $g(x)\to M$ as $x \to x_0$ (and c is a constant), then
$f(x)+g(x)\to L+M$; $f(x)-g(x)\to L-M$; $cf(x)\to cL$;
$f(x)g(x)\to LM$; and $f(x)/g(x)\to L/M$ provided $M \neq 0$

If $f(x)$ is a polynomial, then $\lim_{x\to x_0} f(x) = f(x_0)$

Basic principle: to solve $\lim_{x\to x_0}$, plug in $x = x_0$!

If (and when) you get 0/0 , try something else! (Factor $(x - x_0)$ out of top and bottom...)

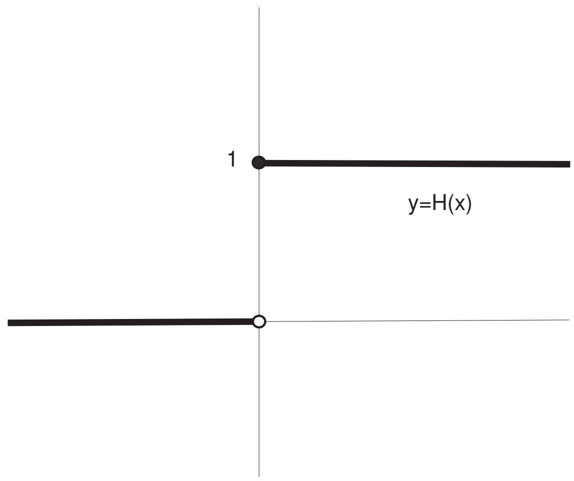
If a function has something like $\sqrt{x} - \sqrt{a}$ in it, try multiplying (top and bottom) with $\sqrt{x} + \sqrt{a}$

Sandwich Theorem: If $f(x)\leq g(x)\leq h(x)$, for all x near a (but not at a), and

$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$, then $\lim_{x\to a} g(x) = L$.

§4: **Extensions of the limit concept**

Motivation: the Heaviside function



The Heaviside function has no limit at 0; it can't make up its mind whether to be 0 or 1. But if we just look to either side of 0, everything is fine; on the left, H(0) `wants' to be 0, while on the right, H(0) `wants' to be 1.

It's because these numbers are different that the limit as we approach 0 does not exist; but the `one-sided' limits DO exist.

Limit from the right: $\lim_{x\to a^+} f(x) = L$ means $f(x)$ is close to L

when x is close to, and <u>bigger</u> than, a

Limit from the left: $\lim_{x\to a^-} f(x) = M$ means $f(x)$ is close to M

when x is close to, and <u>smaller</u> than, a

$\lim_{x\to a} f(x) = L$ then means $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = L$

Infinite limits: ∞ represents something bigger than any number we can think of

$\lim_{x\to a} f(x) = \infty$ means $f(x)$ gets really large as x gets close to a

Also have $\lim_{x\to a} f(x) = -\infty$; $\lim_{x\to a^+} f(x) = \infty$;

$\lim_{x\to a^-} f(x) = \infty$; etc....

Typically, an infinite limit occurs where the denomenator of $f(x)$ is zero (although not always)

§5: **Continuity**

A function f is <u>continuous</u> (cts) <u>at</u> <u>a</u> if $\lim_{x\to a} f(x) = f(a)$

This means: (1) $\lim_{x\to a} f(x)$ exists ; (2) $f(a)$ exists ; and

(3) they're equal.

Limit theorems say (sum, difference, product, quotient) of cts functions are cts.

Polynomials are continuous at every point;

rational functions are continuous except where denom=0.

Points where a function is not continuous are called <u>discontinuities</u>

Four flavors:

removable: both one-sided limits are the same

jump: one-sided limts exist, not the same

infinite: one or both one-sided limits is ∞ or $-\infty$

oscillating: one or both one-sided limits DNE

Intermediate Value Theorem:

If $f(x)$ is cts at every point in an interval $[a, b]$, and M is between $f(a)$ and $f(b)$, then there is (at least one) c between a and b so that $f(c) = M$.

Application: finding roots of polynomials

§6: **Tangent lines**

Slope of tangent line = limit of slopes of secant lines; at $(x_0, f(x_0)$:

$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$ Notation: call this limit $f'(x_0)$ = derivative of f at x_0

Different formulation: $h = x - x_0$, $x = x_0 + h$

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

Chapter 2: Derivatives

§1: **The derivative of a function**

derivative = limit of difference quotient (two flavors)

$f'(x_0)$ exists, say f is <u>differentiable</u> at x_0

Fact: f differentiable (diff'ble) at x_0, then f cts at x_0

$h \to 0$ notation: replace x_0 with x (= variable), get $f'(x)$ = <u>new</u> <u>function</u>

$f'(x)$ = the derivative of f = function whose vaules are the slopes of the tangent lines to the graph of y=$f(x)$. Domain = every point where the limit exists

Notation:

$$f'(x) = \frac{dy}{dx} = \frac{d}{dx}(f(x)) = \frac{df}{dx} = y' = D_x f = Df = (f(x))'$$

§2: **Differentiation rules**

$\frac{d}{dx}$(constant) $= 0$

$\frac{d}{dx}(x) = 1$

$(f(x)$+$g(x))' = (f(x))'$+ $(g(x))'$

$(f(x)$-$g(x))'$= $(f(x))'$- $(g(x))'$

(c$f(x))'$= c$(f(x))'$

$(f(x)g(x))'$= $(f(x))'g(x)$+ $f(x)(g(x))'$

$(\frac{f(x)}{g(x)})'$= $\frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$

$(x^n)'$= nx^{n-1} , for n=0,1,-1,2,-2,3,.......

(($(1/g(x))'$= -ggp/$(g(x))^2$))

$f'(x)$ is 'just' a <u>function</u>, so we can take its derivative!

$(f'(x))'$= $f''(x)$ $\quad (= y'' = \frac{d^2y}{dx^2} = \frac{d^2f}{dx^2})$

= second derivative of f (=rate of change of rate of change of f !)

Keep going! $f'''(x)$ = 3rd derivative, $f^{(n)}(x)$ = nth derivative

§3: **Rates of change**

Physical interpretation:

$f(t)$= position at time t

$f'(t)$= rate of change of position = velocity

$f''(t)$= rate of change of velocity = acceleration

$|f'(t)|$ = speed

Basic principle: for object to change direction (velocity changes sign),

$f'(t)$= 0 somewhere (IVT!)

Examples:

Free-fall: object falling near earth; $s(t) = s_0 + v_0 t - \frac{g}{2}t^2$

$s_0 = s(0)$ = initial position; v_0 = initial velocity; g= acceleration due to gravity

Economics:

$C(x)$ = cost of making x objects; $R(x)$ = revenue from selling x objects;

$P = R - C$ = profit

$C'(x)$ = marginal cost = cost of making 'one more' object

$R'(x)$ = marginal revenue ; profit is maximized when $P'(x) = 0$;

i.e., $R'(x) = C'(x)$

§4: **Derivatives of trigonometric functions**

Basic limit: $\lim_{x\to 0} \frac{\sin x}{x} = 1$; everything else comes from this!

Note: this uses radian measure! $\lim_{x\to 0} \frac{\sin(bx)}{x} = b$

Then we get:

$(\sin x)' = \cos x$ $(\cos x)' = -\sin x$

$(\tan x)' = \sec^2 x$ $(\cot x)' = = \csc^2 x$

$(\sec x)' = \sec x \tan x$ $(\csc x)' = = \csc x \cot x$

§5: **The Chain Rule**

Composition $(g \circ f)(x_0) = g(f(x_0))$; (note: we <u>don't</u> know what $g(x_0)$ is.)

$(g \circ f)'$ ought to have something to do with $g'(x)$ and $f'(x)$

in particular, $(g \circ f)'(x_0)$ should depend on $f'(x_0)$ and $g'(f(x_0))$

Chain Rule: $(g \circ f)'(x_0) = g'(f(x_0))f'(x_0)$

= (d(outside) eval'd at inside fcn)$\cdot$(d(inside))

Ex: $((x^3 + x - 1)^4)' = (4(x^3 + 1 - 1)^3)(3x^2 + 1)$

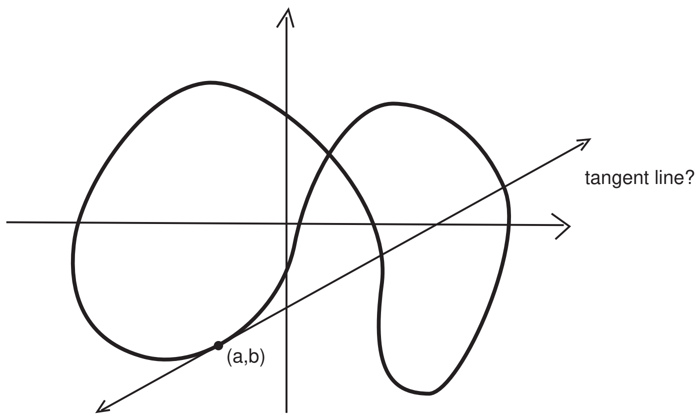
Different notiation:

$y = g(f(x)) = g(u)$, where $u = f(x)$, then $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

§6: **Implicit differentiation**

We can differentiate functions; what about *equations*? (e.g., $x^2 + y^2 = 1$)

graph <u>looks</u> <u>like</u> it has tangent lines



Idea: Pretend equation defines y as a function of x : $x^2+(f(x))^2 = 1$ and differentiate!

$$2x + 2f(x)f'(x) = 0 \text{ ; so} \qquad f'(x) = \frac{-x}{f(x)} = \frac{-x}{y}$$

Different notation:

$$x^2 + xy^2 - y^3 = 6 \text{ ; then } 2x + (y^2 + x(2y\frac{dy}{dx}) - 3y^2\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-2x - y^2}{2xy - 3y^2}$$

Application: extend the power rule

$\frac{d}{dx}(x^r) = rx^{r-1}$ works for any *rational* number r

§7: **Related Rates**

Idea: If two (or more) quantities are related (a change in one value means a change in others), then their rates of change are related, too.

$xyz = 3$; pretend each is a function of t, and differentiate (implicitly).

General procedure:

Draw a picture, describing the situation; label things with variables.

Which variables, rates of change do you know, or want to know?

Find an equation relating the variables whose *rates of change* you know or want to know.

Differentiate!

Plug in the values that you know.