Math 107 Project: Balancing on the point of a pin

Assigned: 2/26/2010 Due: 4/16/2010

This project explores the mathematics behind and applicatons of the *center of mass* (or *center of gravity*) of an object. In many physical situations, an object behaves as if all of its mass were concentrated at a single point, called the center of mass of the object. For example, an object allowed to rotate freely will rotate around a line through its center of mass; an object thrown through the air, in the absence of air resistance, will have its center of mass trace out the perfect parabolic arc that physics predicts. See, for example,

http://www.schooltube.com/video/ef4699826e6448bf9703/Elmo-Center-of-Mass

for experiments carried out with an Elmo doll! In this project we will focus on center of mass computations for an object modeled as a thin plate of uniform density shaped like a region R in the plane; under these hypotheses, the center of mass is usually called the *centroid* of the region R.

For a region R in the xy-plane having a line of reflection symmetry, the centroid will always lie along this line, a fact which can greatly simplify calculations of centroids. Knowlege of the centroid of a region, in turn, can greatly simplify other calculations; the Thoerem of Pappus states that when a region R of the plane is rotated in space around a line not meeting R, the volume of the resulting solid of revolution is equal to the area of R times the distance traveled by the centroid R under the rotation. Our goal is to verify these observations and carry out a variety of computations.

Some basic material on centers of mass can be found in section 6.7 of our text, pages 437-442, which makes a good starting point for your studies. But be **aware**, the notation in the text is not rigorous, whereas the notation of this project is very rigorous. To simplify our work, we begin our study with a one dimensional object like a rod lying on the x-axis.

Part I: One-dimensional objects.

Assume that we have a system of n discrete masses m_k along the x-axis, each located at the coordinate x_k. The moment of each mass m_k is defined to be $m_k x_k$. The moment of the system about the origin is $M_0 = \sum_{k=1}^n m_k x_k$ and the total mass of the system is $M = \sum_{k=1}^n m_k$. The center of mass of this discrete system is defined by the point whose x-coordinate is $\overline{x}$, where

$$\overline{x} = \frac{\sum_{k=1}^n m_k x_k}{\sum_{k=1}^n m_k}.$$

The underlying physical intuition is that since (as you can check) $\sum_{k=1}^n m_k(\overline{x} - x_k) = 0$, where $(\overline{x} - x_k)$ is interpreted as the "signed" distance from x_k to $\overline{x}$, the system of masses will "balance" (neither tip to the right nor to the left) at the center of mass. This is essentially the principle of the lever; a small mass far from the balance point can balance a larger mass close to the balance point but on the other side.

Your first task is to extend this notion to a solid rod of varying density.

Task 1: Consider a rod of length L meters lying on the interval $[0, L]$ on the x-axis. Assume the rod's density is non-constant and given by $\rho(x)$ kg/m, $x \in [0, L]$. Your first task is to show that the center of mass of the rod is

$$\overline{x} = \frac{\int_0^L x\rho(x)dx}{\int_0^L \rho(x)dx}.$$

Idea: Partition the interval $[0, L]$ via the regular partition $\{0 = x_0, x_1, x_2, \cdots, x_n = L\}$, with $\Delta x = \frac{L}{n}$. Now, think of each piece of the rod lying on the kth segment $[x_{k-1}, x_k]$ as a discrete mass whose coordinate is any point of your choice, $z_k \in [x_{k-1}, x_k]$. Approximate $\overline{x}$ as a quotient of two

Riemann sums, and let $n \to \infty$.

Task 2: Use your results in **Task 1** to find the center of mass of a 2-meter rod lying on the interval $[0, 2]$ whose density is given by $\rho(x) = .01\sqrt{x+1}$ kg/m.

Part II: Two-dimensional objects.
Here, we extend the ideas developed for one-dimensional objects to find the center of mass (centroid) of a thin plate occupying a region R. To simplify the problem, we will assume the density of the plate is a constant, say ρ kg/m^2.

As in the one-dimensional case, suppose we have a discrete system of n masses m_k each located at a point (x_k, y_k) in the plane. We define M_x, the moment of the system about the x-axis by:

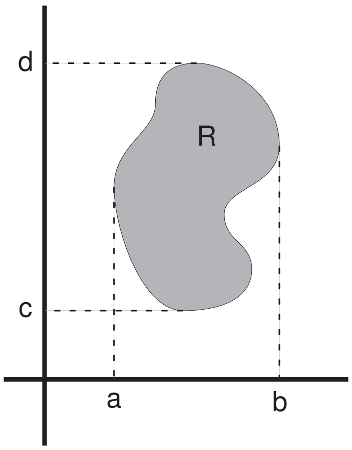
$$M_x = \sum_{k=1}^{n} m_k y_k.$$

Similarly, we define M_y, the moment of the system about the y-axis by: $M_y = \sum_{k=1}^{n} m_k x_k$. Also, the total mass of the system is given by $M = \sum_{k=1}^{n} m_k$. Finally, we define the center of mass of this discrete system to be the point $(\overline{x}, \overline{y})$, where

$$\overline{x} = \frac{M_y}{M} = \frac{\sum_{k=1}^{n} m_k x_k}{\sum_{k=1}^{n} m_k}, \quad \overline{y} = \frac{M_x}{M} = \frac{\sum_{k=1}^{n} m_k y_k}{\sum_{k=1}^{n} m_k}.$$

The intuition is, as before, that $\overline{x} - x_i$ represents the "signed" distance from the point (x_i, y_i) to the line $x = \overline{x}$; the condition $\sum m_i(\overline{x} - x_i) = 0$ (which follows, as before, from the formula above) ensures that the masses, if placed on a massless plate supported along the vertical line $x = \overline{x}$, will balance. The other condition ensures that the masses balance when supported along the horizontal line $y = \overline{y}$. The masses will therefore balance on the point of a pin placed at the center of mass: they will not tip left, right, "up" or "down".

Task 3: Your next task is to fill in the details behind the following computation. Assume we have a thin plate occupying a region R as shown. Also, assume the density of the plate is a constant ρ kg/m^2.



In order to find the centroid of the plate, we start by finding $\overline{x}$. We partition the interval $[a, b]$ via the regular partition $\{a = x_0, x_1, x_2, \cdots, x_n = b\}$, with $\Delta x = \frac{b-a}{n}$. This process results in dividing the plate into thin vertical strips which can be approximated as a rectangle of a small width Δx. Let $L(z_k)$ be the total length of the line segments of intersection of the vertical line $x = z_k$ with R, where $z_k \in [x_{k-1}, x_k]$ is any point of your choice. Now, we think of each vertical strip of the plate as a discrete mass in the plane whose coordinate is (z_k, w_k), for some $w_k \in \mathbb{R}$, which is irrelevant in the following calculations. Let us note that the mass of the kth vertical strip is given by: $m_k =$ (density)(area) $\approx \rho L(z_k)\Delta x$. So, by thinking of the whole plate as a discrete system of n masses $m_k \approx \rho L(z_k)\Delta x$ each located at a point (z_k, w_k) in the plane, we find

$$\overline{x} = \frac{M_y}{M} \approx \frac{\sum_{k=1}^{n} \rho z_k L(z_k)\Delta x}{\sum_{k=1}^{n} \rho L(z_k)\Delta x} = \frac{\sum_{k=1}^{n} z_k L(z_k)\Delta x}{\sum_{k=1}^{n} L(z_k)\Delta x}.$$

By letting $n \to \infty$, we obtain the formula $\overline{x} = \dfrac{\int_a^b xL(x)dx}{\int_a^b L(x)dx}$.

Your task here is to fill in the details explaining why the formula for $\overline{x}$ is valid. Also, you should carry out similar steps to obtain

$$\overline{y} = \frac{\int_c^d yS(y)dy}{\int_c^d S(y)dy},$$

where $S(w_k)$ is the total length of the line segments of intersection of the horizontal line $y = w_k$ with R.

Task 4: Explain why $A(R)$, the area of the region R, is given by: $A(R) = \int_a^b L(x)\,dx = \int_c^d S(y)\,dy$. Hence, we have

$$\overline{x} = \frac{1}{A(R)} \int_a^b xL(x)dx, \quad \overline{y} = \frac{1}{A(R)} \int_c^d yS(y)dy.$$

Use this to explain why, if the region R has a vertical line of reflection symmetry $x = A$, then $\overline{x} = A = \frac{a+b}{2}$, and if R has a horizontal line of reflection symmetry $y = B$, then $\overline{y} = B$. [Hint: a line of symmetry tells us something about the functions $L(x)$ or $S(y)$.]

By computing $L(x)$ and $S(y)$ for specific examples, together with symmetry considerations, we can compute the centroids of a wide variety of regions in the plane:

Task 5: Compute the centroid of a thin plate occupying:
(a): the disk $D = \{(x, y) \ : \ (x + 2)^2 + y^2 \leq 1\}$;
(b): the triangle with vertices $(1, 0)$, $(5, 0)$, and $(4, 4)$;
(c): the region lying between the parabolas $y = 2x - x^2$ and $y = 2x^2 - 4x$

Computations of centroids, especially by symmetry considerations, can aid us in other computations. For example, using the formula for the volume of a solid obtained by revolving a region R around the line $x = c$, by cylindrical shells,

$$\begin{aligned} \text{volume} &= \int_a^b 2\pi|x - c|L(x)\,dx = \pm \int_a^b 2\pi(x - c)L(x)\,dx \\ &= \pm 2\pi(\textstyle\int_a^b xL(x)\,dx - c\int_a^b L(x)\,dx) = \pm 2\pi(\overline{x} - c)A(R) = 2\pi|\overline{x} - c|A(R) \end{aligned}$$

and there is a similar computation for lines $y = c$. This establishes the *Theorem of Pappus*: the volume of a solid of revolution (a region R revolved around an axis in the plane which misses R) is equal to the area of the region R, $A(R)$, times $2\pi|\overline{x} - c|$ (or, for horizontal lines, $2\pi|\overline{y} - c|$), the circumference of the circle traced out by the centroid of R.

Task 6: Use Pappus' Theorem to compute the volumes of the solids obtained by revolving each of the regions in Task 5 around the lines

(a): $x = -3$ (b): $y = 6$.