This project explores the mathematics behind and applications of the center of mass (or center of gravity) of an object. In many physical situations, an object behaves as if all of its mass were concentrated at a single point, called the center of mass of the object. For example, an object allowed to rotate freely will rotate around a line through its center of mass; an object thrown through the air, in the absence of air resistance, will have its center of mass trace out the perfect parabolic arc that physics predicts. See, for example, http://www.schooltube.com/video/ef4699826e6448bf9703/Elmo-Center-of-Mass for experiments carried out with an Elmo doll! In this project we will focus on center of mass computations for an object modeled as a thin plate of uniform density shaped like a region \( R \) in the plane; under these hypotheses, the center of mass is usually called the centroid of the region \( R \).

Part I: One-dimensional objects.
Assume that we have a system of \( n \) discrete masses \( m_k \) along the \( x \)-axis, each located at the coordinate \( x_k \). The moment of each mass \( m_k \) is defined to be \( m_k x_k \). The moment of the system about the origin is \( M_0 = \sum_{k=1}^{n} m_k x_k \) and the total mass of the system is \( M = \sum_{k=1}^{n} m_k \). The center of mass of this discrete system is defined by the point whose \( x \)-coordinate is \( \bar{x} \), where

\[
\bar{x} = \frac{\sum_{k=1}^{n} m_k x_k}{\sum_{k=1}^{n} m_k}.
\]

The underlying physical intuition is that since (as you can check) \( \sum_{k=1}^{n} m_k (\bar{x} - x_k) = 0 \), where \( (\bar{x} - x_k) \) is interpreted as the “signed” distance from \( x_k \) to \( \bar{x} \), the system of masses will “balance” (neither tip to the right nor to the left) at the center of mass. This is essentially the principle of the lever; a small mass far from the balance point can balance a larger mass close to the balance point but on the other side.

Your first task is to extend this notion to a solid rod of varying density.

**Task 1:** Consider a rod of length \( L \) meters lying on the interval \([0, L]\) on the \( x \)-axis. Assume the rod’s density is non-constant and given by \( \rho(x) \) \( \text{kg/m} \), \( x \in [0, L] \). Your first task is to show that the center of mass of the rod is

\[
\bar{x} = \frac{\int_0^L x \rho(x) \, dx}{\int_0^L \rho(x) \, dx}.
\]

**Idea:** Partition the interval \([0, L]\) via the regular partition \{0 = x_0, x_1, x_2, \ldots, x_n = L\}, with \( \Delta x = \frac{L}{n} \). Now, think of each piece of the rod lying on the \( k \)-th segment \([x_{k-1}, x_k]\) as a discrete mass whose coordinate is any point of your choice, \( z_k \in [x_{k-1}, x_k] \). Approximate \( \bar{x} \) as a quotient of two
Riemann sums, and let \( n \to \infty \).

**Task 2:** Use your results in **Task 1** to find the center of mass of a 2-meter rod lying on the interval \([0, 2]\) whose density is given by \( \rho(x) = 0.01\sqrt{x + 1} \text{ kg/m} \).

**Part II: Two-dimensional objects.**
Here, we extend the ideas developed for one-dimensional objects to find the center of mass (centroid) of a thin plate occupying a region \( R \). To simplify the problem, we will assume the density of the plate is a constant, say \( \rho \text{ kg/m}^2 \).

As in the one-dimensional case, suppose we have a discrete system of \( n \) masses \( m_k \) each located at a point \((x_k, y_k)\) in the plane. We define \( M_x \), the moment of the system about the \( x \)-axis by:

\[
M_x = \sum_{k=1}^{n} m_k y_k.
\]

Similarly, we define \( M_y \), the moment of the system about the \( y \)-axis by: \( M_y = \sum_{k=1}^{n} m_k x_k \). Also, the total mass of the system is given by \( M = \sum_{k=1}^{n} m_k \). Finally, we define the center of mass of this discrete system to be the point \((\bar{x}, \bar{y})\), where

\[
\bar{x} = \frac{M_y}{M} = \frac{\sum_{k=1}^{n} m_k x_k}{\sum_{k=1}^{n} m_k}, \quad \bar{y} = \frac{M_x}{M} = \frac{\sum_{k=1}^{n} m_k y_k}{\sum_{k=1}^{n} m_k}.
\]

The intuition is, as before, that \( \bar{x} - x_i \) represents the “signed” distance from the point \((x_i, y_i)\) to the line \( x = \bar{x} \); the condition \( \sum m_i(\bar{x} - x_i) = 0 \) (which follows, as before, from the formula above) ensures that the masses, if placed on a massless plate supported along the vertical line \( x = \bar{x} \), will balance. The other condition ensures that the masses balance when supported along the horizontal line \( y = \bar{y} \). The masses will therefore balance on the point of a pin placed at the center of mass: they will not tip left, right, “up” or “down”.

**Task 3:** Your next task is to fill in the details behind the following computation. Assume we have a thin plate occupying a region \( R \) as shown. Also, assume the density of the plate is a constant \( \rho \) \text{ kg/m}^2.

![Diagram of a thin plate with a region R and axes a to b, c to d, and R labeled](image)

In order to find the centroid of the plate, we start by finding \( \bar{x} \). We partition the interval \([a, b]\) via the regular partition \( \{a = x_0, x_1, x_2, \ldots, x_n = b\} \), with \( \Delta x = \frac{b-a}{n} \). This process results in dividing the plate into thin vertical strips which can be approximated as a rectangle of a small width \( \Delta x \). Let \( L(z_k) \) be the total length of the line segments of intersection of the vertical line \( x = z_k \) with \( R \), where \( z_k \in [x_{k-1}, x_k] \) is any point of your choice. Now, we think of each vertical
strip of the plate as a discrete mass in the plane whose coordinate is \((z_k, w_k)\), for some \(w_k \in \mathbb{R}\), which is irrelevant in the following calculations. Let us note that the mass of the kth vertical strip is given by: \(m_k = (\text{density})(\text{area}) \approx \rho L(z_k)\Delta x\). So, by thinking of the whole plate as a discrete system of \(n\) masses \(m_k \approx \rho L(z_k)\Delta x\) each located at a point \((z_k, w_k)\) in the plane, we find

\[
\bar{x} = \frac{M_y}{M} \approx \frac{\sum_{k=1}^{n} \rho z_k L(z_k) \Delta x}{\sum_{k=1}^{n} \rho L(z_k) \Delta x} = \frac{\sum_{k=1}^{n} z_k L(z_k) \Delta x}{\sum_{k=1}^{n} L(z_k) \Delta x}.
\]

By letting \(n \to \infty\), we obtain the formula

\[
\bar{x} = \frac{\int_{a}^{b} xL(x)dx}{\int_{a}^{b} L(x)dx},
\]

Your task here is to fill in the details explaining why the formula for \(\bar{x}\) is valid. Also, you should carry out similar steps to obtain

\[
\bar{y} = \frac{\int_{c}^{d} yS(y)dy}{\int_{c}^{d} S(y)dy},
\]

where \(S(w_k)\) is the total length of the line segments of intersection of the horizontal line \(y = w_k\) with \(R\).

**Task 4:** Explain why \(A(R)\), the area of the region \(R\), is given by: \(A(R) = \int_{a}^{b} L(x)\ dx = \int_{c}^{d} S(y)\ dy\). Hence, we have

\[
\bar{x} = \frac{1}{A(R)} \int_{a}^{b} xL(x)dx, \quad \bar{y} = \frac{1}{A(R)} \int_{c}^{d} yS(y)dy.
\]

Use this to explain why, if the region \(R\) has a vertical line of reflection symmetry \(x = A\), then \(\bar{x} = A = \frac{a+b}{2}\), and if \(R\) has a horizontal line of reflection symmetry \(y = B\), then \(\bar{y} = B\). [Hint: a line of symmetry tells us something about the functions \(L(x)\) or \(S(y)\).]

By computing \(L(x)\) and \(S(y)\) for specific examples, together with symmetry considerations, we can compute the centroids of a wide variety of regions in the plane:

**Task 5:** Compute the centroid of a thin plate occupying:

(a): the disk \(D = \{(x, y) : (x + 2)^2 + y^2 \leq 1\}\);
(b): the triangle with vertices \((1, 0)\), \((5, 0)\), and \((4, 4)\);
(c): the region lying between the parabolas \(y = 2x - x^2\) and \(y = 2x^2 - 4x\)

Computations of centroids, especially by symmetry considerations, can aid us in other computations. For example, using the formula for the volume of a solid obtained by revolving a region \(R\) around the line \(x = c\), by cylindrical shells,

\[
\text{volume} = \int_{a}^{b} 2\pi |x - c| L(x)\ dx = \pm \int_{a}^{b} 2\pi (x - c) L(x)\ dx
\]

\[
= \pm 2\pi \left( \int_{a}^{b} xL(x)\ dx - c \int_{a}^{b} L(x)\ dx \right) = \pm 2\pi (\bar{x} - c) A(R) = 2\pi |\bar{x} - c| A(R)
\]

and there is a similar computation for lines \(y = c\). This establishes the **Theorem of Pappus**: the volume of a solid of revolution (a region \(R\) revolved around an axis in the plane which misses \(R\)) is equal to the area of the region \(R\), \(A(R)\), times \(2\pi|\bar{x} - c|\) (or, for horizontal lines, \(2\pi|\bar{y} - c|\)), the circumference of the circle traced out by the centroid of \(R\).

**Task 6:** Use Pappus’ Theorem to compute the volumes of the solids obtained by revolving each of the regions in Task 5 around the lines

(a): \(x = -3\) \quad (b): \(y = 6\).