Math 107

Topics for the third exam (and beyond)

(Technically, everything covered on the first two exams plus...)

Polar coordinates

Idea: describe points in the plane in terms of (distance,direction).

$r = (x^2 + y^2)^{1/2} =$ distance , $\theta = \arctan(y/x) =$ angle with the positive x-axis.
$x = r\cos\theta$, $y = r\sin\theta$

The same point in the plane can have many representations in polar coordinates:
$(1,0)_{rect} = (1,0)_{pol} = (1,2\pi)_{pol} = (1,16\pi)_{pol} = \ldots$

A negative distance is interpreted as a positive distance in the *opposite* direction (add π to the angle):
$(-2,\pi/2)_{pol} = (2,\pi/2+\pi)_{pol} = (0,-2)_{rect}$

An equation in polar coordinates can (in principal) be converted to rectangular coords, and vice versa:
E.g., $r = sin(2\theta) = 2\sin\theta\cos\theta$ can be expressed as
$r^3 = (x^2+y^2)^{3/2} = 2(r\sin\theta)(r\cos\theta) = 2yx$, i.e., $(x^2+y^2)^3 = 4x^2y^2$

Given an equation in polar coordinates
$r = f(\theta)$, i.e., the curve $(f(\theta),\theta)_{pol}, \theta_1 \leq \theta \leq \theta_2$
we can compute the slope of its tangent line, by thinking in rectangular coords:
$x = f(\theta)\cos\theta, y = f(\theta)\sin\theta$, so

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}$$

Arclength: the polar curve $r = f(\theta)$ is really the (rectangular) parametrized curve
$x = f(\theta)\cos\theta, y = f(\theta)\sin\theta$, and $(x'(\theta))^2 + (y'(\theta))^2)^{1/2} = (f'(\theta))^2 + (f(\theta))^2)^{1/2}$,
so the arclength for $a \leq \theta \leq b$ is $\displaystyle\int_a^b (f'(\theta))^2 + (f(\theta))^2)^{1/2}\,d\theta$

Area: if $r = f(\theta)$, $a \leq \theta \leq b$ describes a closed curve ($f(a) = f(b) = 0$), then we can compute the area inside the curve as a sum of areas of sectors of a circle, each with area approximately

$$\pi r^2(\Delta\theta/2\pi) = \frac{(f(\theta))^2}{2}\Delta\theta$$

so the area can be computed by the integral $\displaystyle\int_a^b \frac{1}{2}(f(\theta))^2\,d\theta$

Chapter 10: Vectors

Vectors

In one-variable calculus, we make a distinction between speed and velocity; velocity has a direction (left or right), while speed doesn't. Speed is the *size* of the velocity. This distinction is even more important in higher dimensions, and motivates the ntion of a *vector*.

Basically, a vector $\vec{v}$ is an arrow pointing *from* one point in the plane (or 3-space or ...) *to* another. A vector is thought of as pointing frm its *tail* to its *head*. If it points from P to Q, we call the vector $\vec{v} = \overrightarrow{PQ}$.

A vector has both a *size* (= length = distance from P to Q) and a *direction*. Vectors that have the same size and point in the same direction are often thought of as the same, even if they have different tails (and heads). Put differently, by picking up the vector and translating it so that its tail is at the origin (0,0), we can identify $\vec{v}$ with a point in the plane, namely its head (x,y), and write $\vec{v} = \langle x,y\rangle$. If $\vec{v}$ goes from (a,b) to (c,d), then we would have $\vec{v}= \langle c-a, d-b\rangle$. The length of $\vec{v} = \langle a,b\rangle$ is then $||\vec{v}|| = \sqrt{a^2+b^2}$.

In 3-space we have three special vectors, pointing in the direction of each coordinate axis (in the plane there are, analogously, two); these are called

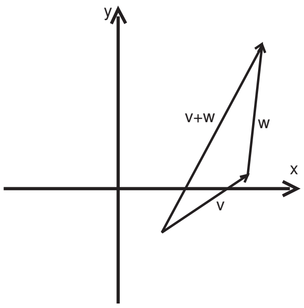
$$\vec{i} = \langle 1,0,0\rangle, \vec{j} = \langle 0,1,0\rangle, \text{ and } \vec{k} = \langle 0,0,1\rangle$$

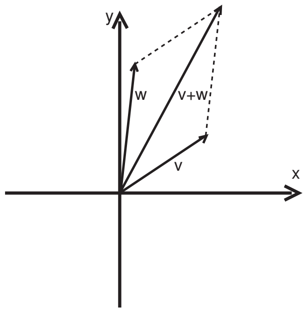
These come in especially handy when we start to add vectors. There are several different points of view to vector addition:

(1) move the vector $\vec{w}$ so that its head is on the tail of $\vec{v}$; then the vector $\vec{v}+\vec{w}$ has tail equal to the tail of $\vec{v}$ and head equal to the head of $\vec{w}$;

(2) move $\vec{v}$ and $\vec{w}$ so that their tails are both at the origin, and build the parallelogram which has sides equal to $\vec{v}$ and $\vec{w}$; then $\vec{v}+\vec{w}$ is the vector that goes from the origin to the opposite corner of the parallelogram;

(3) if $\vec{v} = \langle a, b\rangle$ and $\vec{w} = \langle c, d\rangle$, then $\vec{v}+\vec{w} = \langle a+c, b+d\rangle$





We can also subtract vectors; if they share the same tail, $\vec{v}-\vec{w}$ is the vector that points from the head of $\vec{w}$ to the head of $\vec{v}$ (so that $\vec{w}+(\vec{v}-\vec{w}) = \vec{v}$). In coordinates, we simply subtract the coordinates.

We can also *rescale* vectors = multiply them by a constant factor; $a\vec{v}$ = vector pointing in the same direction, but a times as long. (We use the convention that if $a<0$, then $a\vec{v}$ points in the *opposite* direction from $\vec{v}$.)

Using coordinates, this means that $a\langle x, y\rangle = \langle ax, ay\rangle$. To distinguish a from the coordinates or the vector, we call a a *scalar*. One consequence of this formula is that $||c\vec{v}|| = |c|\cdot||\vec{v}||$.

All of these operations satisfy all of the usual properties you would expect:

$\vec{v}+\vec{w} = \vec{w}+\vec{v}$
$(\vec{v}+\vec{w})+\vec{u} = \vec{v}+(\vec{w}+\vec{u})$
$a(b\vec{v}) = (ab)\vec{v}$
$a(\vec{v}+\vec{w}) = a\vec{v} + a\vec{w}$

If all that we are interested in about a vector is its *direction*, then we can choose a vector of length one pointing in the same direction:

$$\vec{u} = \frac{\vec{v}}{||\vec{v}||} = \text{unit vector pointing in the same diection as } \vec{v} \ .$$

Of course there is nothing special in all of this about vectors in the plane; all of these ideas work for vectors in 3-space. The only thing we really need to determine is the right formula for *length*: a few applications of the Pythagorean theorem leads us to

$$||\langle a, b, c\rangle|| = (a^2+b^2+c^2)^{1/2}$$

Dot products

One thing we haven't done yet is multiply vectors together. It turns out that there are two ways to reasonably do this, serving two very different sorts of purposes. We will do one of them, the *dot product* (or *scalar product*).

The dot product is intended to measure the extent to which two vectors $\vec{v}$ and $\vec{w}$ are pointing in the same direction. It takes a pair of vectors $\vec{v} = \langle v_1, \dots, v_n\rangle$ and $\vec{w} = \langle w_1, \dots, w_n\rangle$, and gives us a *scalar* $\vec{v}\bullet\vec{w} = v_1w_1+\cdots+v_nw_n$.

Note that $\vec{v}\bullet\vec{v} = v_1^2+\cdots+v_n^2 = ||\vec{v}||^2$. In general, $\vec{v}\bullet\vec{w} = ||\vec{v}||\cdot||\vec{w}||\cdot\cos(\theta)$, where θ is the angle between the vectors $\vec{v}$ and $\vec{w}$ (when they have the same tail); this can be seen by comparing the Law of Cosines to the formula

$$||\vec{v}-\vec{w}||^2 = ||\vec{v}||^2+||\vec{w}||^2-2\vec{v}\bullet\vec{w}$$

This in turn allows us to compute this angle:

The *angle* Θ between v and w = the angle (between 0 and π with $\cos(\Theta) = \langle v, w\rangle/(||v||\cdot||w||)$

The dot product satisfies some properties which justify calling it a product:
$\vec{v} \bullet \vec{w} = \vec{w} \bullet \vec{v}$
$(k\vec{v}) \bullet \vec{w} = k(\vec{v} \bullet \vec{w})$
$\vec{v} \bullet (\vec{w} + \vec{u}) = \vec{v} \bullet \vec{w} + \vec{v} \bullet \vec{u}$

Two vectors are orthogonal (= perpendicular) if the angle θ between them is $\pi/2$, so $\cos(\theta)$=0; this means that $\vec{v} \bullet \vec{w} = 0$. We write $\vec{v} \perp \vec{w}$.

Since $|\cos\theta| \leq 1$, we always have $|\vec{v} \bullet \vec{w}| \leq ||\vec{v}||||\vec{w}||$. This is the *Cauchy-Schwartz inequality.* From this we can also deduce the *Triangle inequality* : $||\vec{v} + \vec{w}|| \leq ||\vec{v}|| + ||\vec{w}||$.

Projecting one vector onto another:

The idea is to figure out how much of one vector $\vec{v}$ points *in* the direction of another vector $\vec{w}$. The dot product measures to what extent they are pointing in the same direction, so it is only natural that it plays a role.

What we wish to do is to write $\vec{v} = c\vec{w} + \vec{u}$, where $\vec{u} \perp \vec{w}$ (i.e., write $\vec{v}$ as the part pointing in the direction of $\vec{w}$ and the part $\perp \vec{w}$). By solving the equation $(\vec{v} - c\vec{w}) \bullet \vec{w} = 0$, we find that $c = (\vec{v} \bullet \vec{w})/(\vec{w} \bullet \vec{w})$.

We write $c\vec{w} = \text{proj}_{\vec{w}}\vec{v} = \dfrac{\vec{v} \bullet \vec{w}}{\vec{w} \bullet \vec{w}}\vec{w} = \dfrac{\vec{v} \bullet \vec{w}}{||\vec{w}||}\dfrac{\vec{w}}{||\vec{w}||}$ = (orthogonal) projection of $\vec{v}$ onto $\vec{w}$.

$\vec{u} = \vec{v} - c\vec{w}$ = the part of $\vec{v}$ perpendicular to $\vec{w}$.

Lines and planes in 3-space

Just as with lines in the plane, we can parametrize lines in space, given a point on the line, P, and the direction $\vec{v}$ that the line is traveling:

$L(t) = (x(t), y(t), z(t)) = P + \vec{v}t = (x_0 + at, y_0 + bt, z_0 + ct)$

This involves a (somewhat arbitrary) parameter t to describe; we can find a more *symmetric* description of the line by determining, for each coordinate, what t is and setting them all equal to one another:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

To determine if and where two lines in space intersect, if we use the parametrized forms, we need to remember that the two lines might pass through that same point at different times, and so we really need to use different names for the parameters:

$P + \vec{v}t = Q + \vec{w}s$

This gives us three equations (each of the three coordinates) with two variables; it therefore usually does not have a solutions. Two lines in 3-space that do not meet are called *skew*. If two lines do meet, then we can treat them much like in the plane; we can, for example, determine the angle at which they meet by computing the angle between their direction vectors $\vec{v}, \vec{w}$.

The shortest distance from a point $Q = (a, b, c)$ to a line $P(t) = P + t\vec{v}$ can be computed by minimizing the distance (squared) function: $d(t) = ||P(t) - Q||^2$; we find that $t = \dfrac{(Q - P) \bullet \vec{v}}{||\vec{v}||^2}$, so $t\vec{v}$ is actually the projection of $Q - P$ in the direction of $\vec{v}$, so the distance between point and line is the length of $(Q - P) - proj_{\vec{v}}(Q - P)$, which by Pythagoras has length squared equal to $||Q - P||^2 - \dfrac{[(Q - P) \bullet \vec{v}]^2}{||\vec{v}||^2}$.

Vector-valued functions

Basic idea: think of a parametric curve in 3-space.

$\vec{r}(t) = (x(t), y(t), z(t))$

If we think of t as time, then what $\vec{r}$ does is give us a point in 3-space at each moment of time. Thinking of $\vec{r}$ as the position of a particle, the particle sweeps out a path or curve, C, in 3-space as time passes.

Example: *lines*; they can be described as having a starting place and a direction they travel, and so can be parametrized by $\vec{r}(t) = P + t\vec{v}$, where P is the starting point and $\vec{v}$ is the direction (for example, the difference of two points lying *along* the line).

Vector function calculus

We can extend the concept of a limit to vector-valued functions by thinking in terms of distance; $\vec{r}(t)$ approaches L as t goes to a if the distance between $\vec{r}(t)$ and L tends to 0. This in turn is the same as insisting that each coordinate function $x(t), y(t), z(t)$ tends to the corresponding corrdinate of L as t goes to a. So in particular, a vector function $\vec{r}(t)$ is *continuous* at a if each of its coordinate functions x, y, z are continuous at a.

When we think of t as time, we can imagine ourselves as travelling along the parametrized curve $\vec{r}(t)$, and so at each point we can make sense of both *velocity* and *acceleration*. Velocity, which is the instantaneous rate of change of position, can be calculated as the limit of the usual difference quotient, using the ideas above; but since limits can really be computed one coordinate at a time, the derivative of $\vec{r}(t) = x(t), y(t), z(t)$ is $\vec{v}(t) = \vec{r}'(t) = x'(t), y'(t), z'(t)$.

Some basic properties:

$(\vec{r} + \vec{s})'(t) = \vec{r}'(t) + \vec{s}'(t)$
$(f(t)\vec{r}(t))' = f'(t)\vec{r}(t) + f(t)\vec{r}'(t)$
$(\vec{r} \bullet \vec{s})'(t) = \vec{r}'(t) \bullet \vec{s}(t) + \vec{r}(t) \bullet \vec{s}'(t)$

Similarly, acceleration can be computed as $\vec{a}(t) = \vec{r}''(t) = x''(t), y''(t), z''(t)$; it is the rate of change of the velocity of $\vec{r}(t)$.

One useful fact: if the length of the velocity (i.e., its *speed*), $||\vec{v}(t)||$ is constant, then $\vec{a}(t)$ is always perpendicular to $\vec{v}(t)$

And speaking of length, we can compute the *length* of a parametrized curve by integrating its speed: the length of the parametrized curve $\vec{r}(t)$, $a \leq t \leq b$, is

$$\text{Length} = \int_a^b ||\vec{v}(t)||\ dt$$

Since vector functions have derivatives, which are also vector functions, they therefore have *antiderivatives*; $\vec{R}(t)$ is the antiderivative of $\vec{r}(t)$ if $\vec{R}'(t) = \vec{r}(t)$. Since derivatives can be computed by taking the derivative of each coordinate function, its antiderivative can be computed by taking the antiderivative of each coordinate. Antiderivatives of vector function play an important role in physics; often we are able to measure the velocity of an oblect over an interval of time; by integrating the velocity, and knowing the object's initial position, we recover the position of the object at every time in the interval.

Arclength in space

Finally, by applying the same reasoning we used to produce a formula for the arclength of a parametrized curve in the plane (by taking a limit of lengths of line segments between a sequence of points on the curve), we arrive at a corresponding formula for the length of a parametrized curve $< x(t), y(t), z(t) >$ from $t = a$ to $t = b$:

$$\text{length} = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\ dt$$

A few more practice problems:

1. Find the (rectangular) equation of the line tangent to the graph of the polar curve
$r = 3\sin\theta - \cos(3\theta)$ at the point where $\theta = \dfrac{\pi}{4}$.

2. Find the length of the polar curve $r = \theta^2$ from $\theta = 0$ to $\theta = 2\pi$.

3. Find the area inside of the graph of the polar curve
$r = \sin(\theta) - \cos(\theta)$ from $\theta = \dfrac{\pi}{4}$ to $\theta = \dfrac{5\pi}{4}$.
[Extra credit: What does this curve look like? (Hint: multiply both sides by r.)]

4. Find the orthogonal projection of the vector $\vec{v}$ = (3,1,2) onto the vector $\vec{w}$ = (-1,4,2) .

5. Find the parametric equation of the line through the points
(1,2,3) and (-1,3,4)