Math 107H Exam 1 Solutions

1. (10 pts. each) Find the following indefinite integrals:

(a) $\int \text{Arcsin}(x)\ dx = (*)$

By parts: $u = \text{Arcsin}(x)$, so $du = \dfrac{1}{\sqrt{1-x^2}}\ dx$, and $dv = dx$, so $v = x$. Then

$(*) = x\text{Arcsin}(x) - \displaystyle\int \frac{x}{\sqrt{1-x^2}}\ dx$; this integral we can do by substitution:

$w = 1-x^2$, so $dw = -2x\ dx$, so $(*) = x\text{Arcsin}(x) + \dfrac{1}{2}\displaystyle\int \frac{du}{\sqrt{u}}\Big|_{u=1-x^2}$

$= x\text{Arcsin}(x) + \dfrac{1}{2}\displaystyle\int u^{1/2}\ du\Big|_{u=1-x^2} = \text{Arcsin}(x) + \dfrac{1}{2}2u^{1/2} + c\Big|_{u=1-x^2}$

$= \text{Arcsin}(x) + \sqrt{1-x^2} + c$

(b) $\displaystyle\int \frac{x^2}{\sqrt{1-x^2}}\ dx = (**)$

By trig substitution: $x = \sin u$, so $dx = \cos u\ du$ and $\sqrt{1-x^2} = \cos u$, so

$(**) = \displaystyle\int \frac{\sin^2 u}{\cos u}\cos u\ du\Big|_{x=\sin u} = \int \sin^2 u\ du\Big|_{x=\sin u}$

$= \displaystyle\int \frac{1}{2}(1 - cos(2u)\ du\Big|_{x=\sin u} = \frac{1}{2}(u - \frac{1}{2}\sin(2u)) + c\Big|_{x=\sin u}$

$= \dfrac{1}{2}u - \dfrac{1}{2}\sin u\cos u + c\Big|_{x=\sin u}$

But if $x = \sin u$, then $u = \text{Arcsin}(x)$, and $\cos u = \sqrt{1-x^2}$, so

$(**) = \dfrac{1}{2}\text{Arcsin}(x) - \dfrac{1}{2}x\sqrt{1-x^2} + c$

2. (15 pts.) Find the following definite integral:

$\displaystyle\int_1^3 \frac{x}{(x+1)(x+5)}\ dx\ (***)$

$\dfrac{x}{(x+1)(x+5)} = \dfrac{A}{x+1} + \dfrac{B}{x+5} = \dfrac{A(x+5)+B(x+1)}{(x+1)(x+5)}$, so we need

$x = A(x+5) + B(x+1)$. Setting $x = -5$, we get $-5 = B(-4)$, so $B = \dfrac{5}{4}$.

Setting $x = -1$, we get $-1 = A(4)$, so $A = -\dfrac{1}{4}$.

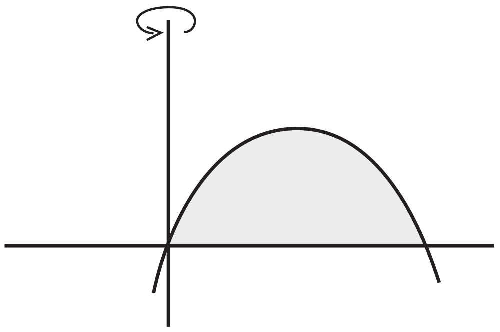
So (***) $= \displaystyle\int_1^3 -\frac{1}{4}\frac{1}{x+1} + \frac{5}{4}\frac{1}{x+5}\ dx$

$= -\dfrac{1}{4}\ln|x+1| + \dfrac{5}{4}\ln|x+5|\Big|_1^3$ (do u-subs to compute each antiderivative)

$= -\dfrac{1}{4}(\ln(4) - \ln(2)) + \dfrac{5}{4}(\ln(8) - \ln(6)) = \dfrac{5}{4}\ln(\dfrac{4}{3}) - \dfrac{1}{4}\ln(2)$

$= \ln((\dfrac{4}{3})^{5/4}(\dfrac{1}{2})^{1/4})$

3. (20 pts.) Find the volume of the region obtained by revolving the region under the graph of $f(x) = \sin x$ from $x = 0$ to $x = \pi$ around the y-axis (see figure).



By cylindrical shells: radius $= x$, height $= \sin x$, so

Volume $= \displaystyle\int_0^\pi 2\pi x \sin x\ dx = 2\pi \int_0^\pi x \sin x\ dx$.

$\displaystyle\int_0^\pi x \sin x\ dx =$ (****) ; integrating by parts,

$u = x$, $dv = \sin x\ dx$, so $du = dx$ and $v = -\cos x$, so

(****) $= -x\cos x - \displaystyle\int -\cos x\ dx = -x\cos x + \int \cos x\ dx = -x\cos x + \sin x + c$

So Volume $= 2\pi(-x\cos x + \sin x)\Big|_0^\pi = 2\pi[(-\pi(-1) + 0) - (0(1) - 0)] = 2\pi^2$.

4. (15 pts.) Find the improper integral $\displaystyle\int_2^\infty \frac{1}{x(\ln x)^3}\ dx$.

$$\int \frac{1}{x(\ln x)^3}\ dx = \int \frac{du}{u^3}\Big|_{u=\ln x}$$ via the u-substitution $u = \ln x$, so $du = \frac{1}{x}\ dx$,

which equals $\int u^{-3}\ du\Big|_{u=\ln x} = -\frac{1}{2}u^{-2} + c\Big|_{u=\ln x} = -\frac{1}{2(\ln x)^2} + c$

So $\int_2^\infty \frac{1}{x(\ln x)^3}\ dx = \lim_{n\to\infty} \int_2^N \frac{1}{x(\ln x)^3}\ dx$

$= \lim_{n\to\infty} -\frac{1}{2(\ln x)^2}\Big|_2^N = \lim_{n\to\infty} \frac{1}{2(\ln 2)^2} - \frac{1}{2(\ln N)^2}$

But since $\ln N \to \infty$ as $N \to \infty$, $\frac{1}{2(\ln N)^2} \to 0$ as $N \to \infty$, so

$$\int_2^\infty \frac{1}{x(\ln x)^3}\ dx = \lim_{n\to\infty} \frac{1}{2(\ln 2)^2} - \frac{1}{2(\ln N)^2} = \frac{1}{2(\ln 2)^2} - 0 = \frac{1}{2(\ln 2)^2}.$$

5. (15 pts.) If we were to compute an approximation to the integral $\int_0^3 \sin(2x)\ dx$ using Simpson's Rule, using $n = 6$ subintervals, how close to the correct answer can we expect our answers to be?

If $I = \int_0^3 \sin(2x)\ dx$, then $|I - S(f,n)| \leq M\frac{(b-a)^5}{180n^4} = M\frac{3^5}{180(6^4)}$, where $|f''''|(x) \leq M$ for every x in the interval $[0,3]$. But

For $f(x) = \sin(2x)$, we have $f'(x) = 2\cos(2x)$, $f''(x) = -4\sin(2x)$, $f'''(x) = -8\cos(2x)$, and $f''''(x) = 16\sin(2x)$,
which, on $[0,3]$ is always (in absolute value) at most 16 (since $|sin(2x) \leq 1$).

So $|I - S(f,n)| \leq 16\frac{3^5}{180(6^4)} = \frac{16\cdot 3^5}{180\cdot 2^4\cdot 3^4} = \frac{3}{180} = \frac{1}{60}$.

$|\int_a^b f(x)\ dx - M(f,n)| \leq K\frac{(b-a)^3}{24n^2}$ $\qquad$ $|\int_a^b f(x)\ dx - T(f,n)| \leq K\frac{(b-a)^3}{12n^2}$

$|\int_a^b f(x)\ dx - S(f,n)| \leq M\frac{(b-a)^5}{180n^4}$

6. (15 pts.) Set up, **but do not evaluate**, the integral which will compute the arclength of the graph of $y = x\sqrt{1+x^2}$ from $x = 0$ to $x = 3$.

$f(x) = x\sqrt{1+x^2} = x(1+x^2)^{\frac{1}{2}}$, so $f'(x) = (1+x^2)^{\frac{1}{2}} + x(\frac{1}{2})(1+x^2)^{-\frac{1}{2}}(2x) = (1+x^2)^{\frac{1}{2}} + x^2(1+x^2)^{-\frac{1}{2}}$.

So Arclength $= \int_0^3 \sqrt{1+[f'(x)]^2}\ dx = \int_0^3 \sqrt{1+[(1+x^2)^{\frac{1}{2}} + x^2(1+x^2)^{-\frac{1}{2}}]^2}\ dx$