Math 107 Practice Exam 2 Solutions

Note: Most sequences/series can be shown to converge or diverge in more than one way; the solutions below illustrate only one such method. Your approach may differ....

1. (10 pts. each) Determine the convergence or divergence of the following sequences:

(a) $a_n = \dfrac{n^3 + 6n^2 \ln n - 1}{2 - 3n^3} = \dfrac{1 + 6(\ln n)/n - 1/n^3}{2/n^3 - 3}$.

and since $1/n^3 \to 0$ and $(\ln n)/n \to 0$ as $n \to \infty$,

$$a_n \to \frac{1 + 6 \cdot 0 - 0}{2 \cdot 0 - 3} = \frac{1}{-3} = -\frac{1}{3} \text{ as } n \to \infty.$$

(b) $b_n = \dfrac{n^{n+\frac{1}{n}}}{(n+3)^n}$

$$b_n = \frac{n^{n+\frac{1}{n}}}{(n+3)^n} = \frac{n^n n^{\frac{1}{n}}}{(n+3)^n} = \frac{n^{\frac{1}{n}}}{\left(\frac{n+3}{n}\right)^n} = \frac{n^{\frac{1}{n}}}{\left(1 + \frac{3}{n}\right)^n}.$$

But $n^{\frac{1}{n}} \to 1$ and $\left(1 + \dfrac{3}{n}\right)^n \to e^3$ as $n \to \infty$, so $b_n \to \dfrac{1}{e^3} = e^{-3}$ as $n \to \infty$.

2. (10 pts. each) Determine the convergence or divergence of the following series:

(a) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{(n-1)(\ln n)^{2/3}}$ [Hint: limit compare, <u>then</u> integral...]

$a_n = \dfrac{1}{(n-1)(\ln n)^{2/3}}$ looks like $b_n = \dfrac{1}{n(\ln n)^{2/3}}$, and $\dfrac{a_n}{b_n} = \dfrac{n}{n-1} \to 1$ as $n \to \infty$, so $\sum a_n$ converges precisely when $\sum b_n$ converges. But:

$b_n = \dfrac{1}{n(\ln n)^{2/3}} = f(n)$ for $f(x) = \dfrac{1}{x(\ln x)^{2/3}}$, which is continuous and decreasing (x and $\ln(x)$ are both increasing, so $(\ln x)^{2/3}$ is increasing, so their reciprocals are decreasing, and so the product is decreasing). So we can apply the integral test:

$\displaystyle\int \frac{1}{x(\ln x)^{2/3}}\, dx = \int \frac{du}{u^{2/3}}\, du|_{u=\ln x} = 3u^{1/3}|_{u=\ln x} = 3(\ln x)^{1/3}$, so

$\displaystyle\int_2^{\infty} \frac{1}{x(\ln x)^{2/3}}\, dx = \lim_{N\to\infty} [3(\ln N)^{1/3} - 3(\ln 2)^{1/3}]$, but since $\ln N \to \infty$ as $N \to \infty$,

$\displaystyle\int_2^{\infty} \frac{1}{x(\ln x)^{2/3}}\, dx$ diverges, so $\sum b_n$ diverges, so $\sum a_n$ **diverges**.

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{6n}{(1-n^2)^2}$ $\quad a_n = \dfrac{6n}{(1-n^2)^2}$ looks like $b_n = \dfrac{6n}{(-n^2)^2} = \dfrac{6}{n^3}$, which converges:

So note that $\dfrac{a_n}{b_n} = \dfrac{(-n^2)^2}{(1-n^2)^2} = \dfrac{1}{(\frac{1}{n^2} - 1)^2}$, and since $1/n^2 \to 0$ as $n \to \infty$, $\dfrac{a_n}{b_n} \to \dfrac{1}{(0-1)^2} = 1$ as $n \to \infty$, so by limit comparison, $\sum a_n$ converges precisely when $\sum b_n$ converges.

But: $\sum b_n = \sum \frac{6}{n^3} = 6\sum \frac{1}{n^3}$, which converges ($p$-series, $p = 3 > 1$), so $\sum b_n$ converges, so $\sum a_n = \sum_{n=0}^{\infty} \frac{6n}{(1-n^2)^2}$ **converges**.

3. (10 pts. each) Determine the convergence or divergence of the following series:

(a) $\sum_{n=1}^{\infty} \frac{(n-1)!}{2^n n^3}$ $\qquad \sum_{n=1}^{\infty} \frac{(n-1)!}{2^n n^3} = \sum a_n$ and

$$\frac{a_{n+1}}{a_n} = \frac{\frac{((n+1)-1)!}{2^{n+1}(n+1)^3}}{\frac{(n-1)!}{2^n n^3}} = \frac{n!}{(n-1)!}\frac{2^n}{2^{n+1}}\frac{n^3}{(n+1)^3} = (n)\left(\frac{1}{2}\right)\left(\frac{n}{n+1}\right)^3.$$

Since $\frac{n}{n+1} \to 1$ and $n \to \infty$ as $n \to \infty$, $\frac{a_{n+1}}{a_n} = (\text{big})(\frac{1}{2})(\text{close to }1)$, which is big, as n gets large, so $\frac{a_{n+1}}{a_n} \to \infty$ as $n \to \infty$, so $\sum a_n$ **diverges** by the Ratio Test.

(b) $\sum_{n=0}^{\infty} \frac{n2^{2n+1}}{9^n+1}$ $\qquad \sum_{n=0}^{\infty} \frac{n2^{2n+1}}{9^n+1} = \sum a_n$ and

$$\frac{a_{n+1}}{a_n} = \frac{\frac{(n+1)2^{2(n+1)+1}}{9^{n+1}+1)}}{\frac{n2^{2n+1}}{9^n+1}} = \frac{(n+1)}{n}\frac{2^{2(n+1)+1}}{2^{2n+1}}\frac{9^n+1}{9^{n+1}+1} = \frac{(1+1/n)}{1}\frac{2^{2n+3}}{2^{2n+1}}\frac{1+9^{-n}}{9+9^{-n}} =$$

$(1+\frac{1}{n})(2^2)\frac{1+9^{-n}}{9+9^{-n}}$, and since $\frac{1}{n} \to 0$ and $9^{-n} \to 0$ as $n \to \infty$,

$$\frac{a_{n+1}}{a_n} = (1+\frac{1}{n})(2^2)\frac{1+9^{-n}}{9+9^{-n}} \to (1+0)(4)\frac{1+0}{9+0} = \frac{4}{9} < 1,$$

so $\sum_{n=0}^{\infty} \frac{n2^{2n+1}}{9^n+1} = \sum a_n$ **converges** by the ratio test.

4. (20 pts.) Compute the radius of convergence of the following power series:

$$f(x) = \sum_{n=0}^{\infty} \frac{2^n-1}{(n+4)^2}(x-3)^n = \sum a_n(x-3)^n$$

$\frac{a_{n+1}}{a_n} = \frac{\frac{2^{n+1}-1}{((n+1)+4)^2}}{\frac{2^n-1}{(n+4)^2}} = \frac{2^{n+1}-1}{2^n-1}\frac{(n+4)^2}{(n+5)^2} = \frac{2-2^{-n}}{1-2^{-n}}\frac{(1+\frac{4}{n})^2}{(1+\frac{5}{n})^2}$, and since

$\frac{1}{n} \to 0$ and $2^{-n} \to 0$ as $n \to \infty$,

$\frac{a_{n+1}}{a_n} = \frac{2-2^{-n}}{1-2^{-n}}\frac{(1+\frac{4}{n})^2}{(1+\frac{5}{n})^2} \to \frac{2-0}{1-0}\frac{(1+4\cdot 0)^2}{(1+5\cdot 0)^2} = 2\cdot 1 = 2 = L$, so the radius of convergence of $\sum a_n(x-3)^n$ is $R = \frac{1}{L} = \frac{1}{2}$.

5. (20 pts.) Find the Taylor polynomial of degree 3, centered at $x = 8$, for the function

$$f(x) = x^{2/3}$$

and estimate the error in using your polynomial to approximate $f(7) = 7^{2/3}$.

To find the Taylor polynomial, we need derivatives:
$f(x) = x^{2/3}$
$f'(x) = (2/3)x^{-1/3}$
$f''(x) = (-1/3)(2/3)x^{-4/3}$
$f'''(x) = (-4/3)(-1/3)(2/3)x^{-7/3}$
Evaluating at $x = 8$, we get
$f(8) = 8^{2/3} = 2^2 = 4$
$f'(8) = (2/3)8^{-1/3} = (2/3)(1/2) = 1/3$
$f''(8) = (-1/3)(2/3)8^{-4/3} = (-1/3)(2/3)2^{-4} = (-2/9)(1/16) = -1/72$
$f'''(8) = (-4/3)(-1/3)(2/3)8^{-7/3} = (8/27)2^{-7} = (1/27)2^{-4} = 1/(27/cdot16)$
So the degree 3 Taylor polynomial is

$$\begin{aligned} P_3(x) = f(8) + f'(8)(x-8) + \frac{f''(8)}{2!}(x-8)^2 + \frac{f'''(8)}{3!}(x-8)^3 \\ = 4 + \frac{1}{3}(x-8) + \frac{-1}{72 \cdot 2}(x-8)^2 + \frac{1}{27 \cdot 16 \cdot 6}(x-8)^3 \end{aligned}$$

For the error term, we need the fourth derivative:
$f''''(x) = (-7/3)(-4/3)(-1/3)(2/3)x^{-10/3}$

We know that the remainder $R_3(x) = f(x) - P_3(x)$ satisfies $|R_3(7)| \leq M\frac{|7-8|^4}{4!}$ where M is the largest value of $|f''''(x)|$ for x between 8 and 7. But $x^{-10/3}$ is a decreasing function, so its largest value will occur at the left endpoint, 7, so

$$|R_3(7)| \leq (-7/3)(-4/3)(-1/3)(2/3)7^{-10/3}\frac{|7-8|^4}{4!} \text{ (whatever that is...)}.$$