Math 107, Section 150

Practice Exam 3 solutions

1. Find the (rectangular) equation of the line tangent to the graph of the polar curve

$$r = 3\sin\theta - \cos(3\theta)$$

at the point where $\theta = \dfrac{\pi}{4}$.

$r = 3\sin\theta - \cos(3\theta) = f(\theta)$, so
$x = r\cos\theta = f(\theta)\cos\theta$ and $y = r\sin\theta = f(\theta)\sin\theta$, so
$dy/dx = \dfrac{dy/d\theta}{dx/d\theta} = \dfrac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}$.
But $f'(\theta) = 3\cos\theta + 3\sin(3\theta)$, so
$f'(\pi/4) = 3\cos(\pi/4) + 3\sin(3\pi/4) = 3(\sqrt{2}/2) + 3(-\sqrt{2}/2) = 0$, and
$f(\pi/4) = 3\sin(\pi/4) - \cos(3\pi/4) = 3(\sqrt{2}/2) - (-\sqrt{2}/2) = 4(\sqrt{2}/2) = 2\sqrt{2}$.
and since $\sin(\pi/4) = \cos(\pi/4) = \sqrt{2}/2$, we have, at $\theta = \pi/4$,

$$dy/dx = \frac{f'(\pi/4)\sin(\pi/4) + f(\pi/4)\cos(\pi/4)}{f'(\pi/4)\cos(\pi/4) - f(\pi/4)\sin(\pi/4)} = \frac{(0)(\sqrt{2}/2) + (2\sqrt{2})(\sqrt{2}/2)}{(0)(\sqrt{2}/2) - (2\sqrt{2})(\sqrt{2}/2)}$$
$$= \frac{(2\sqrt{2})(\sqrt{2}/2)}{-(2\sqrt{2})(\sqrt{2}/2)} = -1.$$

So the slope of the tangent line is -1, and it goes through the point
$(x, y) = (f(\pi/4)\cos(\pi/4), f(\pi/4)\sin(\pi/4)) = ((2\sqrt{2})(\sqrt{2}/2), (2\sqrt{2})(\sqrt{2}/2)) = (2, 2)$,
so the equation for the tangent line is $y - 2 = (-1)(x - 2)$, or $y = -x + 4$.

2. Find the length of the polar curve $r = \theta^2$ from $\theta = 0$ to $\theta = 2\pi$.

For $r = \theta^2 = f(\theta)$, since
$(dx/d\theta)^2 + (dy/d\theta)^2 = (f(\theta))^2 + (f'(\theta))^2 = (\theta^2)^2 + (2\theta)^2 = \theta^4 + 4\theta^2 = \theta^2(\theta^2 + 4)$,
we have

$$\text{Length} = \int_0^{2\pi} \sqrt{\theta^2(\theta^2+4)}\ d\theta = \int_0^{2\pi} \sqrt{\theta^2}\sqrt{\theta^2+4}\ d\theta$$
$$= \int_0^{2\pi} |\theta|\sqrt{\theta^2+4}\ d\theta = \int_0^{2\pi} \theta\sqrt{\theta^2+4}\ d\theta$$

Setting $u = \theta^2 + 4$, then $du = 2\theta\ d\theta$, and for
$\theta = 0$, $u = 4$, while for $\theta = 2\pi$, $u = 4\pi^2 + 4$, so

$$\text{Length} = \frac{1}{2}\int_4^{4\pi^2+4} \sqrt{u}\ du = (\frac{1}{2})(\frac{2}{3})u^{3/2}\Big|_4^{4\pi^2+4}$$
$$= \frac{1}{3}((4\pi^2+4)^{3/2} - 4^{3/2}) = \frac{8}{3}((\pi^2+1)^{3/2} - 1)$$

3. Find the area inside of the graph of the polar curve

$$r = \sin(\theta) - \cos(\theta)$$

from $\theta = \frac{\pi}{4}$ to $\theta = \frac{5\pi}{4}$.

[Extra credit: What does this curve look like? (Hint: multiply both sides by r.)]

Since Area$=\int \frac{1}{2}(f(\theta))^2\ d\theta$, we have

$$\text{Area} = \frac{1}{2}\int_{\pi/4}^{5\pi/4} (\sin\theta - \cos\theta)^2\ d\theta = \frac{1}{2}\int_{\pi/4}^{5\pi/4} \sin^2\theta - 2\sin\theta\cos\theta + \cos^2\theta\ d\theta$$
$$= \frac{1}{2}\int_{\pi/4}^{5\pi/4} 1 - 2\sin\theta\cos\theta\ d\theta = \frac{1}{2}\int_{\pi/4}^{5\pi/4} 1 - \sin(2\theta)\ d\theta = \frac{1}{2}[\theta + \frac{1}{2}\cos(2\theta)]\Big|_{\pi/4}^{5\pi/4}$$
$$= \frac{1}{2}[(5\pi/4 + \frac{1}{2}\cos(5\pi/2)) - (\pi/4 + \frac{1}{2}\cos(\pi/2))] = \frac{1}{2}[(5\pi/4 + \frac{1}{2}) - (\pi/4 + \frac{1}{2})]$$
$$= \frac{1}{2}[(5\pi/4) - (\pi/4)] = \frac{1}{2}[\pi] = \frac{\pi}{2}$$

To see what this curve is, we have $r = \sin(\theta) - \cos(\theta)$, so $r^2 = r\sin(\theta) - r\cos(\theta)$, so $x^2 + y^2 = y - x$, so $(x^2 + x) + (y^2 - y) = 0$, so $(x^2 + x + \frac{1}{4}) + (y^2 - y + \frac{1}{4}) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$, so $(x + \frac{1}{2})^2 + (y - \frac{1}{2})^2 = \frac{1}{2} = (\frac{1}{\sqrt{2}})^2$

This is a circle, centered at $(-\frac{1}{2}, \frac{1}{2})$, with radius $\frac{1}{\sqrt{2}}$!

4. Find the orthogonal projection of the vector $\vec{v}$ = (3,1,2) onto the vector $\vec{w}$ = (-1,4,2) .

The orthogonal projection of $\vec{v}$ onto $\vec{w}$ is

$$\frac{\vec{v} \bullet \vec{w}}{\vec{w} \bullet \vec{w}}\vec{w} = \frac{(3,1,2) \bullet (-1,4,2)}{(-1,4,2) \bullet (-1,4,2)}(-1,4,2) = \frac{-3+4+4}{1+16+4}(-1,4,2)$$
$$= \frac{5}{21}(-1,4,2) = (-\frac{5}{21}, \frac{20}{21}, \frac{10}{21})$$

5. Find the parametric equation of the line through the points

(1,2,3) and (-1,3,4)

The direction vector for the line points from (1,2,3) to (-1,3,4) (or the reverse!), so we may take $\vec{v} =< -1-1, 3-2, 4-3) >=< -2, 1, 1 >$.

So a parametric equation for the line is given by

$$\vec{r}(t) =< 1,2,3 > +t < -2,1,1 >=< 1-2t, 2+t, 3+t > .$$