Math 107H

Another approach to Simpson's Rule

We can justify the formula for Simpson's Rule by trying to imagine a formula which 'cancels out' the errors in the Midpoint Rule and the Trapezoid Rule, but this requires you to know the formulasfor the errors! A different approach, which leads to the same "rule", comes directly from a computation of the integral of a quadratic function.

For $f(x) = ax^2 + bx + c$, $\displaystyle\int_\alpha^\beta f(x)\ dx = \frac{a}{3}x^3 + \frac{b}{2}x^2 + cx\Big|_\alpha^\beta$

$= (\frac{a}{3}\beta^3 + \frac{b}{2}\beta^2 + c\beta) - (\frac{a}{3}\alpha^3 + \frac{b}{2}\alpha^2 + c\alpha) = \frac{a}{3}(\beta^3 - \alpha^3) + \frac{b}{2}(\beta^2 - \alpha^2) + c(\beta - \alpha)$

$= (\beta - \alpha)[\frac{a}{3}(\alpha^2 + \alpha\beta + \beta^2) + \frac{b}{2}(\alpha + \beta) + c] = (\beta - \alpha)Q$, where

$Q = \frac{a}{3}(\alpha^2 + \alpha\beta + \beta^2) + \frac{b}{2}(\alpha + \beta) + c$.   Comparing the formula for Q with

$X = f(\alpha) = a\alpha^2 + b\alpha + c$, $Z = f(\beta) = a\beta^2 + b\beta + c$, and

$Y = f(\frac{\alpha+\beta}{2}) = a\frac{\alpha+\beta}{2})^2 + b\frac{\alpha+\beta}{2} + c = \frac{a}{4}\alpha^2 + \frac{a}{2}\alpha\beta + \frac{a}{4}\beta^2 + \frac{b}{2}\alpha + \frac{b}{2}\beta + c$

and noting that these expressions have the same sorts of terms as Q does, we might expect that Q can be written as some combination of X, Y, and Z. And it can be! Since only Y contains the term $\alpha\beta$ that appears in Q, to capture the $\frac{a}{3}\alpha\beta$ that appears in Q using the $\frac{a}{2}\alpha\beta$ that is in Y, we should "use" $\frac{2}{3}Y$ in our combination. Then:

$Q - \frac{2}{3}Y = \frac{a}{3}\alpha^2 + \frac{a}{3}\alpha\beta + \frac{a}{3}\beta^2 + \frac{b}{2}\alpha + \frac{b}{2}\beta + c - \frac{a}{6}\alpha^2 - \frac{a}{3}\alpha\beta - \frac{a}{6}\beta^2 - \frac{b}{3}\alpha - \frac{b}{3}\beta - \frac{2}{3}c$

$= \frac{a}{6}\alpha^2\frac{a}{6}\beta^2 + \frac{b}{6}\alpha + \frac{b}{6}\beta + \frac{1}{3}c = \frac{1}{6}(a\alpha^2 + b\alpha + c) + \frac{1}{6}(a\beta^2 + b\beta + c) = \frac{1}{6}X + \frac{1}{6}Z$ (!)

So, $Q = \frac{1}{6}X + \frac{4}{6}Y + \frac{1}{6}Z = \frac{1}{6}(f(\alpha) + 4f(\frac{\alpha+\beta}{2}) + f(\beta)$, and so

$$\int_\alpha^\beta ax^2 + bx + c\ dx = \int_\alpha^\beta f(x)\ dx = \frac{\beta-\alpha}{6}[f(\alpha) + 4f(\frac{\alpha+\beta}{2}) + f(\beta)]\ .$$

This formula is the basis for Simpson's Rule; an integral is approximated by a sum which uses the right-hand side of this formula, summed over all subintervals you have cut your original interval into. The point is that this gives the exactly correct answer for quadratic functions; it therefore, in principle, takes into account the concavity of the quadratic, and so, by analogy, should "take into account" the concavity of any function f. This gives, with little extra computational work, a better approximation to the integral of f than formulas which <u>don't</u> "take into account" concavity (like the Midpoint and Trapezoidal rules).

By design, Simpson's Rule gives the exactly correct answer (and not just an approximation) when f is a quadratic function. It is a remarkable fact that is <u>also</u> gives the exact answer when f is a cubic! This is basically because for $f(x) = ax^3$ we have

$$\frac{\beta-\alpha}{6}[f(\alpha) + 4f(\frac{\alpha+\beta}{2}) + f(\beta)] = \frac{a}{6}(\beta-\alpha)[\alpha^3 + 4(\frac{\alpha+\beta}{2})^3 + \beta^3]$$
$$= \frac{a}{6}(\beta-\alpha)[\alpha^3 + \frac{1}{2}(\alpha^3 + 3\alpha^2\beta + 3\alpha\beta^2 + \beta^3) + \beta^3] = \frac{a}{6}(\beta-\alpha)\frac{3}{2}(\alpha^3 + \alpha^2\beta + \alpha\beta^2 + \beta^3)$$
$$= \frac{3a}{12}(\beta-\alpha)(\alpha^3 + \alpha^2\beta + \alpha\beta^2 + \beta^3) = \frac{a}{4}(\beta^4 - \alpha^4) = \int_\alpha^\beta f(x)\ dx\ .$$

(<u>Why</u> does this happen? I don't know...)