Math 107H, Section 1

Quiz number 9 Solutions

Show all work. How you get your answer is just as important, if not more important, than the answer itself.

11. *Use the integral test* to decide if the following series converges:

$$\sum_{n=0}^{\infty} \frac{n}{e^n}$$

[Don't forget to check that we *can* use the integral test!]

We have $a_n = \dfrac{n}{e^n} = f(n)$ for $f(x) = xe^{-x}$. Since $x \geq 0$ and $e^{-x} \geq 0$ whenever $x \geq 0$ we have $f(x) \geq 0$ for all $x \geq 0$. Since $f'(x) = e^{-x} + x(-e^{-x}) = (1-x)e^{-x} < 0$ at least for $x > 1$, f is (eventually) decreasing, and so we can use the integral test. Then:

$$\int xe^{-x}\, dx = x(-e^{-x}) - \int -e^{-x}\, dx = -xe^{-x} + \int e^{-x}\, dx = -xe^{-x} - e^{-x} + c,$$

by applying integration by parts (and u-substitution) $u = x$, $dv = e^{-x}\, dx$, so $du = dx$ and $v = -e^{-x}$. So:

$$\begin{aligned}\int_0^\infty xe^{-x}\, dx &= \lim_{N\to\infty} \int_0^N xe^{-x}\, dx = \lim_{N\to\infty} -xe^{-x} - e^{-x}\Big|_0^N \\ &= \lim_{N\to\infty} (-Ne^{-N} - e^{-N}) - (-0e^{-0} - e^{-0}) \\ &= \lim_{N\to\infty} -\frac{N}{e^N} - \frac{1}{e^N} + 1 = -0 - 0 + 1 = 1,\end{aligned}$$

since $e^N \to \infty$ as $N \to \infty$ and (by L'Hôpital) $\dfrac{N}{e^N} \to 0$

(since this limit is the same as the limit of $\dfrac{(N)'}{(e^N)'} = \dfrac{1}{e^N}$).

So $\displaystyle\int_0^\infty xe^{-x}\, dx$ is a convergent improper integral, and so, by the Integral Test,

$\displaystyle\sum_{n=0}^{\infty} \frac{n}{e^n}$ converges.