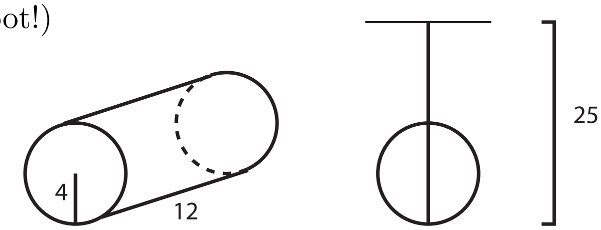
Math 107H, Section 1

Quiz number 8 Solution(s)

Show all work. How you get your answer is just as important, if not more important, than the answer itself.

10. A cylindrical tank is buried on its side so that its bottom is 25 ft deep (see figure). The 'height' of the cyclinder is 12 ft and the 'radius' of the cylinder is 4 ft. If the cyclinder is full of oil, having a weight-density of 50 lb/ft^3, set up, but do not evaluate, the integral which will compute how much work is needed to lift the oil to the surface.

    (Note: weight-density is force per cubic foot!)



To set up the integral, we need coordinates, and there are lots of choices. If we set $x = 0$ to be the center of the circle, then the surface is at $x = 21$. Cross sections of the cylinder are rectangles, with length 12 and width the width of the circle at 'height' x, which since the circle is given by the equation (note that 'x' is the 'second' coordinate!) $a^2 + x^2 = 4^2 = 16$, has width $= 2a = 2\sqrt{4^2 - x^2}$. So we need to lift the volume $(12)(2\sqrt{4^2 - x^2})\ dx$ a distance $21 - x$, where x runs from -4 to 4. With a weight-density of 50, the work done is then

$$\int_{-4}^{4} [(50)(12)(2\sqrt{4^2 - x^2})][21 - x]\ dx = \int_{-4}^{4} 1200(21 - x)\sqrt{16 - x^2}\ dx$$

If we instead put $x = 0$ at ground level, we will lift slices a distance $-x$, where x runs from -25 to -17. The circle is then the circle of radius 4 with center $(0, -21)$ (note that the second coordinate is the x-coord!), so has equation $a^2 + (x + 21)^2 = 16$, so $a = \pm\sqrt{16 - (x + 21)^2}$. So the cross-section of the cylinder has area $(12)(2\sqrt{16 - (x + 21)^2})$, giving us the integral

$$\int_{-25}^{-17} [(50)(12)(2\sqrt{16 - (x + 21)^2})][-x]\ dx = \int_{-25}^{-17} -1200x\sqrt{16 - (x + 21)^2)}\ dx$$

which is the same integral as the first, after the u-substitution $u = x - 21$ ...!

Or! You might set $x = 0$ at the bottom of the tank, in which case you integrate from $x = 0$ to $x = 8$, your cross-sectional area is $(12)(2\sqrt{16 - (x - 4)^2})$, and you lift that slice a distance $25 - x$. This gives the integral

$$\int_{0}^{8} [(50)(12)(2\sqrt{16 - (x - 4)^2})][25 - x]\ dx = \int_{0}^{8} 1200(25 - x)\sqrt{16 - (x - 4)^2)}\ dx$$

which, again, is the same integral, after the u-substitution $u = x + 4$.

Of all of these, the first is probably the least painful one to actually integrate. Splitting it up,

$$\int_{-4}^{4}[1200(21-x)\sqrt{16-x^2}\,dx = \int_{-4}^{4}[(1200)(21)\sqrt{16-x^2}\,dx - \int_{-4}^{4} 1200x\sqrt{16-x^2}\,dx$$

But!

$$\int_{-4}^{4}\sqrt{16-x^2}\,dx$$

is the area of a semicircle of radius 4 (i.e, $(1/2)\pi(4)^2 = 8\pi$), and

$$\int_{-4}^{4} x\sqrt{16-x^2}\,dx$$

is the integral of an odd function over the interval $[-4, 4]$, and so is 0 (!). So the total work done is $(1200)(21)(8\pi)$ foot-pounds.