Math 107H, Section 1

Quiz number 5 Solution

Show all work. How you get your answer is just as important, if not more important, than the answer itself.

Find the following antiderivative:

7. $\displaystyle\int \frac{dx}{x^2\sqrt{4-x^2}} = (*)$.

$\sqrt{4-x^2}$ suggests $x = 2\sin u$ will be useful; then $4-x^2 = 4\cos^2 u$ and $dx = 2\cos u\ du$, and so

$$(*) = \int \frac{2\cos u\ du}{(2\sin u)^2\sqrt{4\cos^2 u}}\Big|_{x=2\sin u} = \frac{2}{2^2\cdot 2}\int \frac{\cos u\ du}{\sin^2 u\cos u}\Big|_{x=2\sin u} = \frac{1}{4}\int \frac{du}{\sin^2 u}\Big|_{x=2\sin u}$$

But $\displaystyle\int \frac{du}{\sin^2 u} = \int \csc^2 u\ du = -\cot u + C$, and so

$$(*) = -\frac{1}{4}\cot u + C\Big|_{x=2\sin u}.$$

But then since $\sin u = \dfrac{x}{2}$, we have $\cos u = \dfrac{\sqrt{4-x^2}}{2}$, and so $\cot u = \dfrac{\sqrt{4-x^2}}{x}$, so

$$\int \frac{dx}{x^2\sqrt{4-x^2}} = -\frac{1}{4}\frac{\sqrt{4-x^2}}{x} + C$$

We can check this answer:

$$\frac{d}{dx}\left(-\frac{1}{4}\frac{\sqrt{4-x^2}}{x}+C\right) = -\frac{1}{4}\frac{(\sqrt{4-x^2})'(x)-(x)'(\sqrt{4-x^2})}{x^2} = -\frac{1}{4}\frac{(\frac{1}{2}\frac{-2x}{\sqrt{4-x^2}})(x)-\sqrt{4-x^2}}{x^2}$$

$$= -\frac{1}{4}\frac{\frac{-x^2}{\sqrt{4-x^2}}-\frac{4-x^2}{\sqrt{4-x^2}}}{x^2} = -\frac{1}{4}\frac{-x^2-(4-x^2)}{x^2\sqrt{4-x^2}} = -\frac{1}{4}\frac{(-x^2-4+x^2)}{x^2\sqrt{4-x^2}} = -\frac{1}{4}\frac{-4}{x^2\sqrt{4-x^2}}$$

$$= \frac{1}{x^2\sqrt{4-x^2}}$$