Math 107H

Topics since the second exam

Note: The final exam will cover everything from the first two topics sheets, as well.

The integral test

Idea: $\displaystyle\sum_{n=1}^{\infty} a_n$ with $a_n \geq 0$ all n, then the partial sums

$\{s_N\}_{N=1}^{\infty}$ forms an increasing sequence;

so converges exactly when bounded from above

If (eventually) $a_n = f(n)$ for a **decreasing** function $f : [a, \infty) \to \mathbf{R}$, then

$$\int_{a+1}^{N+1} f(x)\ \mathrm{d}x \leq s_N = \sum_{n=a}^{N} a_n \leq \int_a^N f(x)\ \mathrm{d}x$$

so $\displaystyle\sum_{n=a}^{\infty} a_n$ converges exactly when $\displaystyle\int_a^{\infty} f(x)\ \mathrm{d}x$ converges

Ex: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges exactly when $p > 1$ (p-series)

Ex: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(\ln n)^p}$ converges exactly when $p > 1$ (logarithmic p-series?)

These families of series make good test cases for comparison with more involved terms (see below!)

Comparison tests

Again, think $\displaystyle\sum_{n=1}^{\infty} a_n$, with $a_n \geq 0$ all n

Convergence depends only on partial sums s_N being **bounded**

One way to determine this: **compare** series with one we **know** converges or diverges

Comparison test: If $b_n \geq a_n \geq 0$ for all n (past a certain point), then

if $\displaystyle\sum_{n=1}^{\infty} b_n$ converges, so does $\displaystyle\sum_{n=1}^{\infty} a_n$; if $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges, so does $\displaystyle\sum_{n=1}^{\infty} b_n$

(i.e., smaller than a convergent series converges; bigger than a divergent series diverges)

More refined: Limit comparison test: a_n and $b_n \geq 0$ for all n, $\dfrac{a_n}{b_n} \to L$

If $L \neq 0$ and $L \neq \infty$, then $\sum a_n$ anf $\sum b_n$ either **both** converge or **both** diverge

If $L = 0$ and $\sum b_n$ converges, then so does $\sum a_n$

If $L = \infty$ and $\sum b_n$ diverges, then so does $\sum a_n$

(Why? eventually $(L/2)b_n \leq a_n \leq (3L/2)b_n$; so can use comparison test.)

Ex: $\sum 1/(n^3 - 1)$ converges; L-comp with $\sum 1/n^3$

$\sum n/3^n$ converges; L-comp with $\sum 1/2^n$

$\sum 1/(n \ln(n^2 + 1))$ diverges; L-comp with $\sum 1/(n \ln n)$

Absolute convergence and alternating series

A series $\sum a_n$ converges absolutely if $\sum |a_n|$ converges. If $\sum |a_n|$ converges then $\sum a_n$ converges. A series which converges but does not converge absolutely is called *conditionally convergent.*

An *alternating series* has the form $\sum(-1)^n a_n$ with $a_n \geq 0$ for all n.

If the sequence a_n is decreasing and has limit 0, then the **alternating series test** states that $\sum(-1)^n a_n$ converges. For example, $\sum_{n=0}^{\infty}(-1)^n/(n+1)$ converges, but not absolutely, so it is conditionally convergent.

Even more, if the alternating series test implies that $\sum(-1)^n a_n$ converges, then the N-th partial sum, $s_N = \sum_{n=0}^{N}(-1)^n a_n$, is within a_{n+1} of the sum of the series (since all of the later partial sums lie between s_N and s_{N+1}).

So, for example, $\sum_{n=1}^{\infty}(-1)^{n+1}/n^2$ converges, and $\sum_{n=1}^{99}(-1)^{n+1}/n^2$ is within $1/(100)^2 = 1/10000$ of the infinite sum. For the series $\sum_{n=1}^{\infty} 1/n^2$, on the other hand, the integral test can only conclude that its tail, $\sum_{n=100}^{\infty} 1/n^2$, is at most $1/100$.

Power series

Idea: turn a series into a function, by making the terms a_n depend on x

replace a_n with $a_n x^n$; series of powers

$$\sum_{n=0}^{\infty} a_n x^n = \text{power series centered at } 0$$

$$\sum_{n=0}^{\infty} a_n (x-a)^n = \text{power series centered at } a$$

Big question: for what x does it converge? Solution from ratio test or root test

$\lim\left|\dfrac{a_{n+1}}{a_n}\right| = L$, or $\lim |a_n|^{\frac{1}{n}} = L$, set $R = \dfrac{1}{L}$

then $\displaystyle\sum_{n=0}^{\infty} a_n(x-a)^n$ converges absolutely for $|x-a| < R$

diverges for $|x-a| > R$; $R =$ radius of convergence

Ex.: $\displaystyle\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$; conv. for $|x| < 1$

Why care about power series?

Idea: partial sums $\displaystyle\sum_{k=0}^{n} a_k x^k$ are polynomials;

if $f(x) = \displaystyle\sum_{n=0}^{\infty} a_n x^n$, then the poly's make good approximations for f

Differentiation and integration of power series

Idea: if you differentiate or integrate each term of a power series, you get a power series which is the derivative or integral of the original one.

If $f(x) = \displaystyle\sum_{n=0}^{\infty} a_n(x-a)^n$ has radius of conv R,

then so does $g(x) = \displaystyle\sum_{n=1}^{\infty} n a_n (x-a)^{n-1}$, and $g(x) = f'(x)$

and so does $g(x) = \displaystyle\sum_{n=0}^{\infty} \frac{a_n}{n+1}(x-a)^{n+1}$, and $g'(x) = f(x)$

Ex: $f(x) = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$, then $f'(x) = f(x)$, so (since $f(0) = 1$) $f(x) = e^x = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$

Ex.: $\dfrac{1}{1-x} = \displaystyle\sum_{n=0}^{\infty} x^n$, so $\quad -\ln(1-x) = \displaystyle\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$ (for $|x|<1$), so

(replacing x with $-x$) $\ln(x+1) = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}$, so

(replacing x with $x-1$) $\ln(x) = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n (x-1)^{n+1}}{n+1}$

Ex:. $\arctan x = \displaystyle\int \frac{1}{1-(-x^2)}\,\mathrm{d}x = \int \sum_{n=0}^{\infty} (-x^2)^n\,\mathrm{d}x =$

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} \qquad \text{(for } |x|<1$$

Taylor series

Idea: start with function $f(x)$, find power series <u>for</u> it.

If $f(x) = \displaystyle\sum_{n=0}^{\infty} a_n (x-a)^n$, then (term by term diff.)

$f^{(n)}(a) = n! a_n$; So $a_n = \dfrac{f^{(n)}(a)}{n!}$

Starting with f, define $P(x) = \displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$,

the Taylor series for f, centered at a.

$P_n(x) = \displaystyle\sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$, the n-th Taylor polynomial for f.

Ex.: $f(x) = \sin x$, then $P(x) = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$

Big questions: Is $f(x) = P(x)$? (I.e., does $f(x) - P_n(x)$ tend to 0 ?)
If so, how well do the P_n's approximate f ? (I.e., how small is $f(x) - P_n(x)$?)

Error estimates

$f(x) = \displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$

means that the value of f at a point x (far from a) can be determined just from the behavior of f near a (i.e., from the derivs. of f at a). This is a very powerful property, one that we wouldn't ordinarily expect to be true. The amazing thing is that it often is:

$T_{f,a}(x) = \displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$; $T_{n,f,a}(x) = \displaystyle\sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^n$;

$R_n(x) = f(x) - T_{n,f,a}(x) = n$-th remainder term = error in using $T_{n,f,a}$ to approximate f

Taylor's remainder theorem : estimates the size of $R_n(x)$

If $f(x)$ and all of its derivatives (up to $n+1$) are continuous on $[a,b]$, then

$f(b) = T_{n,f,a}(b) + \displaystyle\int_a^b \frac{f^{(n+1)}(t)}{(n+1)!}(b-t)^n$.

How? By starting from $f(b) = f(a) + \displaystyle\int_a^b f'(t)\,dt$, and repeatedly integrating by parts *the 'wrong' way*! This in turn implies:

$f(b) = T_{n,f,a}(b) + \dfrac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}$, for some c in $[a,b]$

i.e., for each x, $R_n(x) = \dfrac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$, for some c between a and x

so if $|f^{(n+1)}(x)| \le M$ for every x in $[a,b]$, then $|R_n(x,a)| \le \dfrac{M}{(n+1)!}(x-a)^{n+1}$ for every x in $[a,b]$

Ex.: $f(x)$=$\sin x$, then $|f^{(n+1)}(x)| \le 1$ for all x, so $|R_n(x,0)| \le \dfrac{|x|^{n+1}}{(n+1)!} \to 0$ as $n \to \infty$

so $\sin x = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1}$ Similarly, $\cos x = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n}$

Using Taylor's remainder to estimate values of functions:

$e^x = \displaystyle\sum_{n=0}^{\infty} \frac{(x)^n}{(n)!}$, so e=e^1=$\displaystyle\sum_{n=0}^{\infty} \frac{1}{(n)!}$

$|R_n(1,0)| = \dfrac{f^{(n+1)}(c)}{(n+1)!} = \dfrac{e^c}{(n+1)!} \le \dfrac{e^1}{(n+1)!} \le \dfrac{4}{(n+1)!}$

since $e < 4$ (since $\ln(4) > (1/2)(1) + (1/4)(2) = 1$)
(Riemann sum for integral of $1/x$)

so since $\dfrac{4}{(13+1)!} = 4.58\times 10^{-11}$,

$e = 1 + 1 + \dfrac{1}{2} + \dfrac{1}{6} + \dfrac{1}{24} + \dfrac{1}{120} + \cdots + \dfrac{1}{13!}$, to 10 decimal places.

Other uses: if you know the Taylor series, it tells you the values of the derivatives at the center.

Ex.: e^x=$\displaystyle\sum_{n=0}^{\infty} \frac{(x)^n}{(n)!}$, so $xe^x = \displaystyle\sum_{n=0}^{\infty} \frac{(x)^{n+1}}{(n)!}$, so

15th deriv of xe^x , at 0, is 15!(coeff of x^{15}) $= \dfrac{15!}{14!} = 15$

Substitutions: new Taylor series out of old ones

Ex. $\sin^2 x = \dfrac{1-\cos(2x)}{2} = \dfrac{1}{2}(1 - \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n(2x)^{2n}}{(2n)!}$

$= \dfrac{1}{2}(1-(1-\dfrac{(2x)^2}{2!} + \dfrac{(2x)^4}{4!} - \dfrac{(2x)^6}{6!} + \cdots$

$= \dfrac{2x^2}{2!} - \dfrac{2^3x^4}{4!} + \dfrac{2^5x^6}{6!} - \dfrac{2^7x^8}{8!} + \cdots$

Integrate functions we can't handle any other way:

Ex.: $e^{x^2} = \displaystyle\sum_{n=0}^{\infty} \frac{(x)^2n}{(n)!}$, so $\displaystyle\int e^{x^2}\, dx = \sum_{n=0}^{\infty} \frac{(x)^{2n+1}}{n!(2n+1)}$

Polar coordinates

Idea: describe points in the plane in terms of (distance,direction).

$r = (x^2+y^2)^{1/2}$ = distance , $\theta = \arctan(y/x)$ = angle with the positive x-axis.
$x = r\cos\theta$, $y = r\sin\theta$

The same point in the plane can have many representations in polar coordinates:

$(1,0)_{rect} = (1,0)_{pol} = (1,2\pi)_{pol} = (1,16\pi)_{pol} = \ldots$

A negative distance is interpreted as a positive distance in the *opposite* direction (add π to the angle):

$(-2,\pi/2)_{pol} = (2,\pi/2+\pi)_{pol} = (0,-2)_{rect}$

An equation in polar coordinates can (in principal) be converted to rectangular coords, and vice versa:

E.g., $r = \sin(2\theta) = 2\sin\theta\cos\theta$ can be expressed as
$r^3 = (x^2+y^2)^{3/2} = 2(r\sin\theta)(r\cos\theta) = 2yx$, i.e., $(x^2+y^2)^3 = 4x^2y^2$

Graphing in polar coordinates: graph $r = f(\theta)$ as if it were Cartesian; this allows us to identify the values of θ (= sectors of the circle) where r is positive/negative and increasing/decreasing (i.e., moving away from/towards the origin). Now wrap the Cartesian graph around the origin, using the values of θ where $f = 0$ and $f' = 0$ as a guide.

Given an equation in polar coordinates
$r = f(\theta)$, i.e., the curve $(f(\theta), \theta)_{pol}, \theta_1 \le \theta \le \theta_2$
we can compute the slope of its tangent line, by thinking in rectangular coords:
$x = f(\theta)\cos\theta, y = f(\theta)\sin\theta$, so

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}$$

Arclength: the polar curve $r = f(\theta)$ is really the (rectangular) parametrized curve
$x = f(\theta)\cos\theta, y = f(\theta)\sin\theta$, and $((x'(\theta))^2 + (y'(\theta))^2)^{1/2} = ((f'(\theta))^2 + (f(\theta))^2)^{1/2}$,
so the arclength for $a \le \theta \le b$ is $\displaystyle\int_a^b ((f'(\theta))^2 + (f(\theta))^2)^{1/2}\, d\theta$

Area: if $r = f(\theta)$, $a \le \theta \le b$ describes a closed curve ($f(a) = f(b) = 0$), then we can compute the area inside the curve as a sum of areas of sectors of a circle, each with area approximately

$$\pi r^2(\Delta\theta/2\pi) = \frac{(f(\theta))^2}{2}\Delta\theta$$

so the area can be computed by the integral $\displaystyle\int_a^b \frac{1}{2}(f(\theta))^2\, d\theta$

For the area between two polar curves: if $f(\theta) \ge g(\theta)$ for $\alpha \le \theta \le \beta$, then

$$\text{Area} = \int_\alpha^\beta \frac{1}{2}(f(\theta))^2 - \frac{1}{2}(g(\theta))^2\, d\theta$$

Fourier series

Idea: a different way to express a function as a sum of 'nicer' functions. The nice functions this time, though, are trig functions (instead of powers)!

The other idea: Taylor series of f is built using information from only around the center $x = a$ of the series. For many functions, though, this tells us nothing (i.e., we get no good approximation) further from a. For example, it can tell us nothing past a point of discontinuity of f. A different approach uses *integration* to capture information 'averaged' over an entire interval.

Starting with a *periodic* function f, with period (for the sake of illustration, any number can be used) 2π, so $f(x + 2\pi) = f(x)$ for every x, the idea is to express f as an (infinite) sum of nice functions, also having period 2π. One natural choice to make is the functions $\sin(nx)$ and $\cos(nx)$, so we will attempt to write

$$f(x) = \sum_{n=0}^{\infty} a_n \sin(nx) + b_n \cos(nx)$$

Two immediate questions: does such a series converge, and can we actually do this?! Usually, yes! Just as with Taylor series, the right question to ask is: what values must a_n and b_n have? The answer is obtained by integration!

Since $\displaystyle\int_{-\pi}^{\pi} \sin(mx)\cos(nx)\, dx = 0$ for all m and n (the integrands are odd functions!), and

$$\int_{-\pi}^{\pi} \sin(mx)\sin(nx)\, dx = 0 \text{ and } \int_{-\pi}^{\pi} \cos(mx)\cos(nx)\, dx = 0 \text{ for } m \ne n,$$

while $\displaystyle\int_{-\pi}^{\pi} \sin(nx)\sin(nx)\ dx = \int_{-\pi}^{\pi} \cos(nx)\cos(nx)\ dx = \frac{\pi}{2}$

(these can be verified by integration by parts!), this means that

$$\int_{-\pi}^{\pi} f(x)\sin(mx)\ dx$$

$$= \sum_{n=0}^{\infty} a_n \int_{-\pi}^{\pi} \sin(nx)\sin(mx)\ dx\ + b_n \int_{-\pi}^{\pi} \cos(nx)\sin(mx)\ dx = \frac{\pi}{2}a_m \text{ and}$$

$$\int_{-\pi}^{\pi} f(x)\cos(mx)\ dx$$

$$= \sum_{n=0}^{\infty} a_n \int_{-\pi}^{\pi} \sin(nx)\cos(mx)\ dx\ + b_n \int_{-\pi}^{\pi} \cos(nx)\cos(mx)\ dx = \frac{\pi}{2}b_m$$

and so $\displaystyle a_n = \frac{2}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\ dx$ and $\displaystyle b_n = \frac{2}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\ dx$.

So *if* we can express a function as an infinite sum of trig functions, this is what the coefficients must be equal to! For example, if we compute this for the "square wave", the function f with $f(x) = -1$ for $x \in [-\pi, 0)$ and $f(x) = 1$ for $x \in [0, \pi)$ (and which then repeats this pattern in both directions), some computation gives us that

$\displaystyle a_n = \frac{2(1-(-1)^n)}{n\pi}$ and $b_n = 0$ (since f is an odd function). Graphing the sums $\displaystyle\sum_{n=0}^{N} \frac{2(1-(-1)^n)}{n\pi}\sin(nx)$ for increasingly large values of N does give a sequence of functions which give good approximations to f !

It is somewhat beyond the scope of our course to verify this, but the theory behind all of this is that the coefficients we have computed succeed in giving the smallest possible value for the integral

$$\int_{-\pi}^{\pi} [f(x) - \sum_{n=0}^{N}(a_n\sin(nx) + b_n\cos(nx))]^2\ dx$$

for every N, that these integrals decrease with N, and (usually!) converge to 0, implying that over <u>most</u> of the interval $[-\pi, \pi]$ the 'error' $\displaystyle\left|f(x) - \sum_{n=0}^{N}[a_n\sin(nx) + b_n\cos(nx)]\right|$ must be small!