Why Lagrange was Right

We computed $\displaystyle\int_0^\pi \sin x\ dx = 2$ directly from the limit of Riemann sums definition, with the help of what is known as (one of) *Lagrange's Trigonometric Identity*(ies):

$$\sum_{i=1}^{n} \sin(ix) = \frac{1}{2}\cot(\frac{x}{2}) - \frac{\cos(n+\frac{1}{2})x}{2\sin(\frac{x}{2})}$$

We can show that this formula is correct (essentially finishing this integral computation):

Noting first that the right-hand side of Lagrange's identity is equal to $\dfrac{\cos(\frac{x}{2}) - \cos(n+\frac{1}{2})x}{2\sin(\frac{x}{2})}$

and setting $u = \dfrac{x}{2}$ (so $x = 2u$), since all of those denominators are annoying, what Lagrange really says is that

$$(*) \quad \sum_{i=1}^{n} \sin(2iu) = \frac{\cos(u) - \cos(2n+1)u}{2\sin(u)}$$

We can show that this is true, using a technique known as *mathematical induction*: we show that (*) holds when $n = 1$, and <u>if</u> (*) holds when $n = N$ then it <u>also</u> holds when $n = N + 1$. So being true for $n = 1$ means it is true for $n = 2$, so it is true for $n = 3$, and so for $n = 4$, and so on and so on!

For starters, we need that (for $n = 1$)

$$\sin(2u) = \frac{\cos(u) - \cos(3u)}{2\sin(u)}, \text{ that is, } 2\sin(u)\sin(2u) = \cos(u) - \cos(3u)$$

But this is a kind of trig identity you might know: setting $A = u$ and $B = 2u$, then using angle sum and difference formulas for cosine, we have

$$\begin{aligned}\cos(u) - \cos(3u) &= \cos(B - A) - \cos(B + A)\\ &= [\cos(B)\cos(A) + \sin(B)\sin(A)] - [\cos(B)\cos(A) - \sin(B)\sin(A)]\\ &= 2\sin(B)\sin(A) = 2\sin(A)\sin(B) = 2\sin(u)\sin(2u)\end{aligned}$$

so (*) is true when $n = 1$. We should remember this fact:

$$(**)\ 2\sin(A)\sin(B) = \cos(B - A) - \cos(B + A)$$

since we will use it again shortly! Now suppose that

$$\sum_{i=1}^{N} \sin(2iu) = \frac{\cos(u) - \cos(2N+1)u}{2\sin(u)}$$

Then:

$$\begin{aligned}\sum_{i=1}^{N+1} \sin(2iu) &= \Big\{\sum_{i=1}^{N} \sin(2iu)\Big\} + \sin(2(N+1)u) = \Big\{\frac{\cos(u) - \cos(2N+1)u}{2\sin(u)}\Big\} + \sin(2N+2)u\\ &= \frac{\cos(u) - [\cos(2N+1)u - 2\sin(u)\sin(2N+2)u]}{2\sin(u)}\end{aligned}$$

What we <u>want</u> this to be equal to is

$$\frac{\cos(u) - \cos(2(N+1)+1)u}{2\sin(u)} = \frac{\cos(u) - \cos(2N+3)u}{2\sin(u)}$$

Comparing the two expressions, to make them equal what we <u>need</u> is that

$$\cos(2N+3)u = \cos(2N+1)u - 2\sin(u)\sin(2N+2)u,$$

that is

$$2\sin(u)\sin(2N+2)u = \cos(2N+1)u - \cos(2N+3)u$$

But that might look kind of familiar! It sounds a lot like the first identity that we established. And, in fact, if we set $A = u$ and $B = (2N+2)u$, this equality that we need reads

$2\sin(A)\sin(B) = \cos(B-A) - \cos(B+A)$,

which <u>is</u> the trig identity that we already established! <u>So</u> we have shown that if (*) holds when $n = N$ then it also holds when $n = N+1$. So our "inductive" step holds, and we have established that (*) holds for every $n = 1, 2, 3, 4, \ldots$. Which is what we needed to establish that $\displaystyle\int_0^\pi \sin x\ dx = 2$ <u>without</u> appealing to the Fundamental Theorem of Calculus....

In the end, once we've seen <u>why</u> it holds, perhaps the 'right' way to view Lagrange's identity is by clearing the denominator:

$$\sum_{i=1}^{n} 2\sin(u)\sin(2iu) = 2\sin(u)\sin(2u) + 2\sin(u)\sin(4u) + \cdots + 2\sin(u)\sin(2nu)$$
$$= [\cos(u) - \cos(3u)] + [\cos(3u) - \cos(5u)] + [\cos(5u) - \cos(7u)] + \cdots + [\cos((2n-1)u) - \cos((2n+1)u)]$$
$$= \cos(u) - [\cos(3u) - \cos(3u)] - [\cos(5u) - \cos(5u)] - [\cos(7u) - \cdots - \cos((2n-1)u)] - \cos((2n+1)u)$$
$$= \cos(u) - 0 - 0 - \cdots - 0 - \cos(2n+1)u = \cos(u) - \cos(2n+1)u\ .$$

The sum, once we have used the trig identity (**), becomes what we will call a 'telescoping' sum; when we rearrange the terms it collapses on itself. This is a concept we will return to (and exploit) in other situations in the future!

This same trig identity can be used to show that $\displaystyle\int_0^{\text{anything}} \sin x\ dx = 1 - \cos(\text{anything})$, by replacing x in the identity with the appropriate replacement for $\pi/(2n)$ (namely, anything$/(2n)$). This then leads to $\displaystyle\int_{\text{blah}}^{\text{bleh}} \sin x\ dx = \cos(\text{bleh}) - \cos(\text{blah})$ (by 'splitting' the interval [blah, bleh] at 0), which is our familiar answer to this integration problem!