Improper integrals: dominant terms decide

When we are dealing with an improper integral of the form $\int_a^\infty \frac{f(x)}{g(x)}\,dx$, our perspective is tht the 'dominant' terms in the numerator and denominator determine whether or not the integral converges. By 'dominant' we mean that if $f(x) = a_1(x)+a_2(x)+\cdots+a_n(x)$, then the term $a_1(x)$ dominates if $\frac{a_i(x)}{a_1(x)} \to 0$ as $x \to \infty$, for every other term $i = 2,\ldots,n$. Then if $a_1(x)$ and $b_1(x)$ are the dominant terms of the top and bottom, our result is that

$$\int_a^\infty \frac{f(x)}{g(x)}\,dx \text{ converges precisely when } \int_a^\infty \frac{a_1(x)}{b_1(x)}\,dx \text{ converges.}$$

One way to justify this in any particular instance is to note that since $\frac{a_i(x)}{a_1(x)} \to 0$ and $\frac{b_i(x)}{b_1(x)} \to 0$ for every $i \geq 2$, when we write

$$f(x) = a_1(x)+a_2(x)+\cdots+a_n(x) = a_1(x)[1+\frac{a_2(x)}{a_1(x)}+\cdots+\frac{a_n(x)}{a_1(x)}] = a_1(x)A(x) \text{ and}$$
$$g(x) = b_1(x)+a_2(x)+\cdots+b_m(x) = b_1(x)[1+\frac{b_2(x)}{b_1(x)}+\cdots+\frac{b_m(x)}{b_1(x)}] = b_1(x)B(x)$$

we have $A(x) \to 1$ and $B(x) \to 1$ as $x \to \infty$ (since, other than the 1 in each sum, every other term goes to 0). This means that <u>eventually</u> $A(x)$ and $B(x)$ are both close to 1, so, eventually, say, $0.8 < A(x) < 1.2$ and $0.8 < B(x) < 1.2$. But then, eventually,

$$0.8a_1(x) < a_1(x)A(x) = f(x) = a_1(x)A(x) < 1.2a_1(x) \text{ and}$$
$$0.8b_1(x) < b_1(x)B(x) = g(x) = b_1(x)B(x) < 1.2b_1(x),$$

[We are supposing here that the dominant terms $a_1(x), b_1(x)$ are <u>positive</u>; if they are not, pull a minus sign out of the entire integral first! Otherwise, the inequalities go the <u>opposite</u> directions....]

and so, eventually,

$$\frac{2}{3}\frac{a_1(x)}{b_1(x)} = \frac{0.8a_1(x)}{1.2b_1(x)} < \frac{f(x)}{1.2b_1(x)} < \frac{f(x)}{g(x)} < \frac{f(x)}{0.8b_1(x)} < \frac{1.2a_1(x)}{0.8b_1(x)} = \frac{3}{2}\frac{a_1(x)}{b_1(x)}\ .$$

So $\frac{2}{3}\frac{a_1(x)}{b_1(x)} < \frac{f(x)}{g(x)} < \frac{3}{2}\frac{a_1(x)}{b_1(x)}$.

This means that if $\int_a^\infty \frac{a_1(x)}{b_1(x)}\,dx$ converges, then so does $\frac{3}{2}\int_a^\infty \frac{a_1(x)}{b_1(x)}\,dx$, and since (eventually) $\int_N^\infty \frac{f(x)}{g(x)}\,dx$ is smaller than $\frac{3}{2}\int_N^\infty \frac{a_1(x)}{b_1(x)}\,dx$, $\int_N^\infty \frac{f(x)}{g(x)}\,dx$ converges, and so $\int_a^\infty \frac{f(x)}{g(x)}\,dx$ converges.

On the other hand, if $\int_a^\infty \frac{a_1(x)}{b_1(x)}\,dx$ diverges, then so does $\frac{2}{3}\int_a^\infty \frac{a_1(x)}{b_1(x)}\,dx$, and since (eventually) $\int_N^\infty \frac{f(x)}{g(x)}\,dx$ is larger than $\frac{2}{3}\int_N^\infty \frac{a_1(x)}{b_1(x)}\,dx$, $\int_N^\infty \frac{f(x)}{g(x)}\,dx$ diverges, and so $\int_a^\infty \frac{f(x)}{g(x)}\,dx$ diverges.

The same basic principle applies for most other situations. For example, dealing with something inside of a square (or other) root:

$\int_2^\infty \frac{x\,dx}{\sqrt{x^5-6x+2}}$ converges, since $\sqrt{x^5-6x+2}=\sqrt{x^5}\sqrt{1-\frac{6}{x^4}+\frac{2}{x^5}}$, and $\sqrt{1-\frac{6}{x^4}+\frac{2}{x^5}} \to 1$ as $x\to\infty$, and so eventually $\frac{x}{2\sqrt{x^5}} < \frac{x}{\sqrt{x^5-6x+2}} < \frac{2x}{\sqrt{x^5}}$, and since $\int_a^\infty \frac{x\,dx}{\sqrt{x^5}} = \int_a^\infty \frac{dx}{x^{3/2}}$ converges, so does the original integral.

Integrals with a limit of integration equal to $-\infty$ behave similarly; we could use a u-substitution $u=-x$ to directly turn it into a integral in the form above.

Improper integrals where the function 'blows up' at an endpoint a (or inside of the interval) also have 'dominant terms', usually determined by the smallest powers of $x-a$ in the numerator and denominator. [To make things less challenging for ourselves, we can use a u-substitution $u=x-a$ (or $u=a-x$) to move the vertical asymptote to $u=0$.] The principle is basically the same:

For example, the function $\frac{x^2-x+2}{\sqrt{x^3+x^5}}$, near $x=0$, behaves like $\frac{2}{\sqrt{x^3}}=\frac{2}{x^{3/2}}$ (whose integral diverges), since

$$f(x)=\frac{x^2-x+2}{\sqrt{x^3+x^5}}=\frac{x^2-x+2}{\sqrt{x^3(1+x^2)}}=\frac{1}{\sqrt{x^3}}\frac{x^2-x+2}{\sqrt{1+x^2}}$$

and $\frac{x^2-x+2}{\sqrt{1+x^2}} \to \frac{2}{1}=2$ as $x\to 0$. So we can (eventually, i.e., near $x=0$) trap $f(x)$ between two multiples of $\frac{1}{\sqrt{x^3}}$, and so the convergence of the integral of f mirrors that of $\frac{1}{\sqrt{x^3}}$.

Generally, if we pull out the smallest powers of numerator and denominator and set them aside, then what remains will converge to a non-zero, finite number as we approach the asymptote. This means that the function behaves like a constant multiple of the pieces we set aside, and so has the same convergence 'profile'.