Trigonometric integrals: the whole story

A trigonometric integral is a (sum of) products or quotients of trig functions. Converting $\tan x$, $\sec x$, etc. to sin's and cos's in the usual way, these integrals then come in 4 flavors:

$$\int \sin^n x\ \cos^m x\ dx,\ \int \frac{\sin^n x}{\cos^m x}\ dx,\ \int \frac{\cos^m x}{\sin^n x}\ dx,\ \int \frac{1}{\sin^n x\ \cos^m x}\ dx$$

For each of these forms, there are (usually at least two!) generally successful methods for "reducing" the integral to something which our other approaches can handle.

The basic underlying ideas are:

(1) Use u-substitution when possible to convert the problem to an integral of a polynomial or rational (= quotient of polynomials) function.

(2) Use the trig identities

$\sin^2 x + \cos^2 x = 1$, $\tan^2 x + 1 = \sec^2 x$, $1 + \cot^2 x = \csc^2 x$ (*)

and $\sin^2 x = \frac{1}{2}(1 - \cos(2x))$, $\cos^2 x = \frac{1}{2}(1 + \cos(2x))$, $\sin x \cos x = \frac{1}{2}\sin(2x)$ (**)

to convert the integrand in order to help us reach an integral on which u-substitution can be successful employed.

More precisely: for $\int \sin^n x\ \cos^m x\ dx$, if n or m is odd, then

$$\int \sin^n x\ \cos^m x\ dx = \int \sin^n x\ \cos^{m-1} x\ (\cos x\ dx) = \int \sin^n x\ (\cos^2 x)^{\frac{m-1}{2}}\ (\cos x\ dx) =$$
$$= \int \sin^n x\ (1 - \sin^2 x)^{\frac{m-1}{2}}\ (\cos x\ dx),\ \underline{\text{or}}$$
$$\int \sin^n x\ \cos^m x\ dx = \int \sin^{n-1} x\ \cos^m x\ (\sin x\ dx) = \int (\sin^2 x)^{\frac{n-1}{2}}\ \cos^m x\ (\sin x\ dx) =$$
$$- \int (1 - \cos^2 x)^{\frac{n-1}{2}}\ \cos^m x\ (-\sin x\ dx)$$

and the appropriate u-substitution will give us a polynomial to integrate.

If both n and m are even, then using the formulas (**) will yield trig integrals (in the "variable" $2x$) whose exponents are smaller; repeated use of these formulas will yield integrals that the first technique can be applied to:

$$\int \sin^{2n} x\ \cos^{2m} x\ dx = \int (\sin^2 x)^n (\cos^2 x)^m\ dx = \int (\frac{1}{2}(1 - \cos(2x)))^n (\frac{1}{2}(1 + \cos(2x)))^m\ dx$$

and multiply out! Odd exponents can be handled the first way; with even exponents we can repeat this process <u>again</u>, obtaining trig integrals in the "variable " $4x$, etc.

For $\int \frac{\sin^n x}{\cos^m x}\ dx$: if $n \leq m$ then we can pair each $\sin x$ with a $\cos x$ to convert this to $\int \tan^n x \sec^{m-n} x\ dx = \int \tan^n x \sec^k x\ dx$.

[If $n > m$ then we can replace one $\sin^2 x$ on top with $1 - \cos^2 x$ to split the integral into two integrals, one with two fewer $\sin x$'s, and the other with two fewer of both $\sin x$ and $\cos x$. Repeated use of this will result in integrals of the forms we want: $n \le m$ (together with possibly an integral of the first type we looked at).]

These integrals can be dealt with in a similar fashion. If k is even (and $k \ge 2$), then

$$\int \tan^n x \sec^k x\ dx = \int \tan^n x \sec^{k-2} x\ (\sec^2 x\ dx) = \int \tan^n x (\sec^2 x)^{\frac{k-2}{2}}\ (\sec^2 x\ dx) = \int \tan^n x (\tan^2 x + 1)^{\frac{k-2}{2}}\ (\sec^2 x\ dx)$$

and the substitution $u = \tan x$ will produce an integral of a polynomial to solve.

If n is odd, then

$$\int \tan^n x \sec^k x\, dx = \int \tan^{n-1} x \sec^{k-1} x\, (\sec x \tan x\ dx) = \int (\tan^2 x)^{\frac{n-1}{2}} \sec^{k-1} x\, (\sec x \tan x\ dx)$$
$$= \int (\sec^2 x - 1)^{\frac{n-1}{2}} \sec^{k-1} x\ (\sec x \tan x\ dx)$$

and the substitution $u = \sec x$ will procude a polynomial to integrate.

If k is odd and n is even, then

$$\int \tan^n x \sec^k x\ dx = \int (\tan^2 x)^{\frac{n}{2}} \sec^k x\ dx = \int (\sec^2 - 1)^{\frac{n}{2}} \sec^k x\ dx$$

which can be multiplied out to produce a sum of integrals of (odd) powers of $\sec x$. Then an integration by parts (integrating $\sec^2 x$, differentiating the rest) together with some algebraic manipulation can be used to construct the **reduction formula**

$$\int \sec^n x\ dx = \frac{1}{n-1} \sec^{n-2} x \tan x + \frac{n-2}{n-1} \int \sec^{n-2} x\ dx$$

This allows us to continually reduce the exponent in our integral, until we reach

$$\int \sec^2 x\ dx = \tan x + C$$

$$\int \sec x\ dx = \int \sec x \frac{\sec x + \tan x}{\sec x + \tan x}\ dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\ dx = \int \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x}\ dx$$
$$= \ln|\sec x + \tan x| + C$$

Finally, if $k = 0$ [and n is even] (i.e., we have an integral of a power of $\tan x$), then using $\tan^2 x = \sec^2 x - 1$ we can convert this to powers of $\sec x$, and apply the previous approach.

For $\displaystyle\int \frac{\cos^m x}{\sin^n x}\ dx$:

We could develop a completely parallel theory, using $\cot x$ and $\csc x$ instead of $\tan x$ and $\sec x$. Or we can use the substitution $u = \frac{\pi}{2} - x$ [so $x = \frac{\pi}{2} - u$] and the fact that

$\sin(\frac{\pi}{2} - u) = \cos u$ and $\cos(\frac{\pi}{2} - u) = \sin u$ to rewrite $\displaystyle\int \frac{\cos^m x}{\sin^n x}\ dx = \int \frac{\sin^m u}{\cos^n u}\ (-du)\Big|_{u=\frac{\pi}{2}-x}$, and apply the previous collection of techniques!

For $\displaystyle\int \frac{1}{\sin^n x\ \cos^m x}\,dx$, we basically just do the best we can to end up somwhere else!
If n or m is odd, then we employ u-substitution:

$$\int \frac{1}{\sin^n x\ \cos^m x}\,dx = \int \frac{\sin x}{\sin^{n+1} x\ \cos^m x}\,dx = \int \frac{\sin x}{(1-\cos^2 x)^{\frac{n+1}{2}} x\ \cos^m x}\,dx$$

or $$\int \frac{1}{\sin^n x\ \cos^m x}\,dx = \int \frac{\cos x}{\sin^n x\ \cos^{m+1} x}\,dx = \int \frac{\cos x}{(\sin^n x)(1-\sin^2 x)^{\frac{m+1}{2}}}\,dx$$

and the appropriate u-substitution will result in the integral of a rational function of u, $\displaystyle\int \frac{du}{u^n(1-u^2)^{\frac{m+1}{2}}}\Big|_{u=\sin x}$ (which we will learn <u>later</u> how to integrate!).

If n and m are both even, then using the identity $\sin x \cos x = \frac{1}{2}\sin(2x)$,
if $n = m$ then we can rewrite the integral as an integral of a power of $\csc(2x)$, which we can apply previous techniques to solve. [How? Just pretend that $m - n = 0$ in the computations below.] If $n > m$ or $n < m$, then we make the exponents in the denominator equal at the expense of putting a term in the numerator, and then convert:

$$\int \frac{1}{\sin^n x\ \cos^m x}\,dx = \int \frac{\sin^{m-n} x}{\sin^m x\ \cos^m x}\,dx = \int \frac{(\sin^2 x)^{\frac{m-n}{2}}}{(\sin x\cos x)^m}\,dx = \int \frac{(\frac{1}{2}(1-\cos(2x)))^{\frac{m-n}{2}}}{(\frac{1}{2}\sin(2x))^m}\,dx$$

or $$\int \frac{1}{\sin^n x\ \cos^m x}\,dx = \int \frac{\cos^{n-m} x}{\sin^n x\ \cos^n x}\,dx = \int \frac{(\cos^2 x)^{\frac{n-m}{2}}}{(\sin x\cos x)^n}\,dx = \int \frac{(\frac{1}{2}(1+\cos(2x)))^{\frac{n-m}{2}}}{(\frac{1}{2}\sin(2x))^n}\,dx$$

which can be treated (after multiplying out! and substitution $u = 2x$) as integrals of $\cos u$'s over $\sin u$'s.

It is worth noting that for the integrals $\displaystyle\int \sin^n x\ \cos^m x\ dx$ with n and m both even, there is another approach which mimics the integrals $\displaystyle\int \tan^n x \sec^k x\ dx$:

Use $\cos^2 x = 1 - \sin^2 x$ to convert all $\cos x$ to $\sin x$ and use a **reduction formula** for powers of $\sin x$ (obtained by parts by integrating a $\sin x$, differentiating the rest, and some algebra involving $\cos^2 x = 1 - \sin^2 x$)

$$\int \sin^n x\ dx = \frac{-1}{n}\sin^{n-1} x\cos x + \frac{n-1}{n}\int \sin^{n-2} x\ dx$$

Summary of formulas:

$$\int \sin^n x\ \cos^m x\ dx = \int \sin^n x\ (1-\sin^2 x)^{\frac{m-1}{2}}\ (\cos x\ dx) \qquad \text{if } m \text{ odd}$$

$$\int \sin^n x\ \cos^m x\ dx = -\int (1-\cos^2 x)^{\frac{n-1}{2}}\ \cos^m x\ (-\sin x\ dx) \qquad \text{if } n \text{ odd}$$

$$\int \sin^{2n} x\ \cos^{2m} x\ dx = \int (\frac{1}{2}(1-\cos(2x)))^n (\frac{1}{2}(1+\cos(2x)))^m\ dx$$

$$\int \sin^n x\ dx = \frac{-1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x\ dx \text{ (reduction formula)}$$

$$\int \tan^n x \sec^k x\ dx = \int (\tan^n x)(\tan^2 x + 1)^{\frac{k-2}{2}}\ (\sec^2 x\ dx) \qquad \text{if } k \geq 2 \text{ is even}$$

$$\int \tan^n x \sec^k x\ dx = \int (\sec^2 x - 1)^{\frac{n-1}{2}} \sec^{k-1} x\ (\sec x \tan x\ dx) \qquad \text{if } n \text{ is odd}$$

$$\int \tan^n x \sec^k x\ dx = \int (\sec^2 - 1)^{\frac{n}{2}} \sec^k x\ dx \qquad \text{if } k \text{ odd, } n \text{ even}$$

$$\int \sec^n x\ dx = \frac{1}{n-1} \sec^{n-2} x \tan x + \frac{n-2}{n-1} \int \sec^{n-2} x\ dx \qquad \text{(reduction formula)}$$

$$\int \sec x\ dx = \ln|\sec x + \tan x| + C$$

$$\int \frac{\cos^m x}{\sin^n x}\ dx = -\int \frac{\sin^m u}{\cos^n u}\ du\Big|_{u=\frac{\pi}{2}-x}$$

$$\int \frac{1}{\sin^n x\ \cos^m x}\ dx = \int \frac{\sin x}{(1-\cos^2 x)^{\frac{n+1}{2}} x\ \cos^m x}\ dx \qquad \text{if } n \text{ odd}$$

$$\int \frac{1}{\sin^n x\ \cos^m x}\ dx = \int \frac{\cos x}{(\sin^n x)(1-\sin^2 x)^{\frac{m+1}{2}}}\ dx \qquad \text{if } m \text{ odd}$$

$$\int \frac{1}{\sin^n x\ \cos^m x}\ dx = \int \frac{[\frac{1}{2}(1-\cos(2x))]^{\frac{m-n}{2}}}{[\frac{1}{2}\sin(2x)]^m}\ dx \qquad \text{if } m \geq n$$

$$\int \frac{1}{\sin^n x\ \cos^m x}\ dx = \int \frac{[\frac{1}{2}(1+\cos(2x))]^{\frac{n-m}{2}}}{[\frac{1}{2}\sin(2x)]^n}\ dx \qquad \text{if } n \geq m$$

$$\int \sin(mx)\ \cos(nx)\ dx = \frac{1}{2}\int \sin(m+n)x + \sin(m-n)x\ dx$$

$$\int \sin(mx)\ \sin(nx)\ dx = \frac{1}{2}\int -\cos(m+n)x + \cos(m-n)x\ dx$$

$$\int \cos(mx)\ \cos(nx)\ dx = \frac{1}{2}\int \cos(m+n)x + \cos(m-n)x\ dx$$