Partial fractions: the integrals

Our discussion of partial fractions centered on how to rewrite a rational function $\frac{p(x)}{q(x)}$ as a sum of "nicer" functions. But how do we integrate the nicer functions? There are two kinds of functions we find ourselves faced with:

$$\frac{A}{(x-r)^k} \text{ and } \frac{Ax+B}{(x^2+ax+b)^k}$$

where x^2+ax+b has no real roots (i.e., $a^2-4b<0$). For the first, a substitution $u=x-r$ will find us needing to integrate $Au^{-k}\ du$, which should not worry us much. For the second, completing the square, so

$$x^2+ax+b=(x+\frac{a}{2})^2-\frac{a^2}{4}+b=\big(x+\frac{a}{2}\big)^2+\big(\frac{\sqrt{4b-a^2}}{2}\big)^2$$

we have, setting $y=x+\frac{a}{2}$ (and, for ease of notation, $c=\frac{\sqrt{4b-a^2}}{2}$

$$\begin{aligned}\frac{Ax+B}{(x^2+ax+b)^k}\ dx &= \frac{A(y-\frac{a}{2})+B}{(y^2+c^2)^k}\ dy = \frac{Ay+(B-\frac{a}{2}A)}{(y^2+c^2)^k}\ dy = \frac{Ay+D}{(y^2+c^2)^k}\ dy \\ &= \frac{Ay}{(y^2+c^2)^k}\ dy + \frac{D}{(y^2+c^2)^k}\ dy\end{aligned}$$

(where $y=x+\frac{a}{2}$ (of course) and $D=B-\frac{a}{2}A$ to clean up the notation again).

The first of these pieces, $\frac{Ay}{(y^2+c^2)^k}\ dy$, again should not worry us; a second substitution $u=y^2+c^2$ will turn this into a power of u, which we can integrate. But what about the second piece? If $k=1$, then this integral is on our list:

$$\int\frac{dy}{y^2+c^2}=\frac{1}{c}\arctan(\frac{y}{c})+E$$

(if you forget this, a trig sub $y=c\tan u$ will give you the integral of $\frac{1}{c}\ du$ to solve).

For $k>1$, we can find a <u>reduction</u> <u>formula</u> by differentiating the function $\frac{y}{(y^2+c^2)^{k-1}}$:

$$\begin{aligned}\frac{d}{dy}\Big(\frac{y}{(y^2+c^2)^{k-1}}\Big) &= \frac{(1)(y^2+c^2)^{k-1}-y(k-1)(y^2+c^2)^{k-2}(2y)}{(y^2+c^2)^{2(k-1)}} \\ &= \frac{1}{(y^2+c^2)^{k-1}}-\frac{2(k-1)y^2}{(y^2+c^2)^k}=\frac{1}{(y^2+c^2)^{k-1}}-\frac{2(k-1)(y^2+c^2)-2(k-1)c^2}{(y^2+c^2)^k} \\ &= \frac{1-2(k-1)}{(y^2+c^2)^{k-1}}+\frac{2(k-1)c^2}{(y^2+c^2)^k}=(3-2k)\frac{1}{(y^2+c^2)^{k-1}}+(2k-2)c^2\frac{1}{(y^2+c^2)^k}\end{aligned}$$

Integrating (and pushing terms around), this becomes:

$$c^2\int\frac{dy}{(y^2+c^2)^k}=\frac{1}{(2k-2)}\cdot\frac{y}{(y^2+c^2)^{k-1}}+\frac{(2k-3)}{(2k-2)}\int\frac{dy}{(y^2+c^2)^{k-1}}$$

Which is a reduction formula!