Finding $\int \ln(1+x^2)\ dx$ the "other" way...

It takes a bit longer, but we can make progress on this integral using the "obvious" substitution:

$u = 1 + x^2$, then $x^2 = u - 1$, $x = \sqrt{u-1}$, and $dx = \dfrac{du}{2\sqrt{u-1}}$, so

$\int \ln(1+x^2)\ dx = \displaystyle\int \frac{\ln(u)}{2\sqrt{u-1}}\ du\Big|_{u=1+x^2}$.

Is this progress? Well, this we can make progress on by parts:

$v = \ln(u)$, $dw = \dfrac{du}{2\sqrt{u-1}}$, then $dv = \dfrac{du}{u}$ and $w = \sqrt{u-1}$ (from the above computation!),

so $\displaystyle\int \frac{\ln(u)}{2\sqrt{u-1}}\ du = \sqrt{u-1}\ln(u) - \int \frac{\sqrt{u-1}}{u}\ du$

But now we can make progress on the resulting integral by substitution again!

For $\displaystyle\int \frac{\sqrt{u-1}}{u}\ du$:

$y = \sqrt{u-1}$, then $dy = \dfrac{du}{2\sqrt{u-1}}$ (we keep coming back to this computation!),
and $y^2 = u - 1$, so $u = y^2 + 1$, and we get

$$\int \frac{\sqrt{u-1}}{u}\ du = \int \frac{2(\sqrt{u-1})^2}{u} \cdot \frac{du}{2\sqrt{u-1}} = \int \frac{2y^2}{y^2+1}\ dy\Big|_{y=\sqrt{u-1}}$$

But! $\displaystyle\int \frac{2y^2}{y^2+1}\ dy = \int \frac{2[(y^2+1)-1]}{y^2+1}\ dy = 2\int 1 - \frac{1}{y^2+1}\ dy = 2[y - \arctan(y)] + C$

So, putting it all together,

$$\int \ln(1+x^2)\ dx = \int \frac{\ln(u)}{2\sqrt{u-1}}\ du\Big|_{u=1+x^2}$$

$$= \sqrt{u-1}\ln(u)\Big|_{u=1+x^2} - \int \frac{\sqrt{u-1}}{u}\ du\Big|_{u=1+x^2} = \sqrt{x^2}\ln(1+x^2) - \int \frac{\sqrt{u-1}}{u}\ du\Big|_{u=1+x^2}$$

$$= |x|\ln(1+x^2) - \int \frac{2y^2}{y^2+1}\ dy\Big|_{y=\sqrt{u-1}}\Big|_{u=1+x^2} = |x|\ln(1+x^2) - \int \frac{2y^2}{y^2+1}\ dy\Big|_{y=\sqrt{x^2}}$$

$$= |x|\ln(1+x^2) - \int \frac{2y^2}{y^2+1}\ dy\Big|_{y=|x|} = |x|\ln(1+x^2) - 2[y - \arctan(y)] + C\Big|_{y=|x|}$$

$$= |x|\ln(1+x^2) - 2|x| + 2\arctan(|x|) + C$$

In summary, we substitute $u = 1 + x^2$, then integrate by parts, <u>then</u> do the <u>reverse</u> substitution $y = \sqrt{u-1}$. The more direct approach (provided as a solution to the quiz) essentally combines all of these together, into a single integration by parts.