Solutions to some of the

Math 107H Practice problems for exam 1

4. $\displaystyle\int \frac{dx}{x\sqrt{x^2+1}} = (*)$.

Substituting $x = \tan u$, we have $dx = \sec^2 u\ du$, and $x^2+1 = \tan^2 u + 1 = \sec^2 u$, so

$$(*) = \int \frac{\sec^2 u\ du}{(\tan u)(\sec u)}|_{x=\tan u} = \int \frac{\sec u\ du}{\tan u}|_{x=\tan u} = \int \sec u \cot u\ du|_{x=\tan u}$$
$$= \int \frac{1}{\cos u}\frac{\cos u}{\sin u}\ du|_{x=\tan u} = \int \frac{1}{\sin u}\ du|_{x=\tan u} = \int \csc u\ du|_{x=\tan u}$$
$$= \ln|\csc u - \cot u| + c|_{x=\tan u}\ .$$

Using the (right) right triangle, $\tan u = \dfrac{x}{1}$, so $\csc u = \dfrac{\sqrt{x^2+1}}{x}$ and $\cot u = \dfrac{1}{x}$, so

$$\int \frac{dx}{x\sqrt{x^2+1}} = \ln|\csc u - \cot u| + c|_{x=\tan u}$$
$$= \ln|\frac{\sqrt{x^2+1}}{x} - \frac{1}{x}| + c = \ln|\frac{\sqrt{x^2+1}-1}{x}| + c = \ln|\sqrt{x^2+1}-1| - \ln|x| + c\ .$$

Alternate approach:

$\displaystyle\int \frac{dx}{x\sqrt{x^2+1}} = \int \frac{x\ dx}{x^2\sqrt{x^2+1}} = (**)$. With $u = x^2+1$, $du = 2x\ dx$, and $x^2 = u-1$, so

$\displaystyle(**) = \frac{1}{2}\int \frac{du}{(u-1)\sqrt{u}}|_{u=x^2+1} = \int \frac{1}{(u-1)}\frac{du}{2\sqrt{u}}|_{u=x^2+1} = (***)$.

Setting $v = \sqrt{u}$, $dv = \dfrac{du}{2\sqrt{u}}$, and $u = v^2$, so $\displaystyle(***) = \int \frac{1}{(v^2-1)}\ dv|_{v=\sqrt{u}}|_{u=x^2+1} = (****)$

But! $\displaystyle\frac{1}{(v^2-1)} = \frac{1}{(v-1)(v+1)} = \frac{A}{v-1} + \frac{B}{v+1} = \frac{A(v+1)+B(v-1)}{(v-1)(v+1)}$

when $A(v+1)+B(v-1) = 1$; plugging in $v=1$ yields $2A = 1$, so $A = 1/2$,
and plugging in $v=-1$ yields $-2B = 1$, so $B = -1/2$. So

$$(****) = \frac{1}{2}\int \frac{1}{(v-1)} - \frac{1}{(v+1)}\ dv|_{v=\sqrt{u}}|_{u=x^2+1} = \frac{1}{2}(\ln|v-1| - \ln|v+1|) + c|_{v=\sqrt{u}}|_{u=x^2+1}$$
$$= \frac{1}{2}(\ln|\sqrt{u}-1| - \ln|\sqrt{u}+1|) + c|_{u=x^2+1}$$
$$= \frac{1}{2}(\ln|\sqrt{x^2+1}-1| - \ln|\sqrt{x^2+1}+1|) + c$$

5. $\displaystyle\int \frac{x^2\ dx}{(x-2)(x^2+1)}$.

$$\frac{x^2}{(x-2)(x^2+1)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+1} = \frac{A(x^2+1)+(Bx+C)(x-2)}{(x-2)(x^2+1)},\text{ so}$$

$x^2 = A(x^2+1) + (Bx+C)(x-2)$ for some A, B, C .

Setting $x=2$, we get $4 = (A)(5) + (2B+C)(0) = 5A$, so $A = \dfrac{4}{5}$.

Setting $x=0$, we get $0 = (A)(1) + (B(0)+C)(-2) = \dfrac{4}{5} - 2C$, so $2C = \dfrac{4}{5}$, so $C = \dfrac{2}{5}$.

Setting $x=1$, we get $1 = (A)(2) + (B(1)+C)(-1) = 2\dfrac{4}{5} - B - \dfrac{2}{5}$, so

$B = 2\frac{4}{5} - \frac{2}{5} - 1 = \frac{8-2-5}{5} = \frac{1}{5}$.

So $\frac{x^2}{(x-2)(x^2+1)} = \frac{4}{5}\frac{1}{x-2} + \frac{1}{5}\frac{x}{x^2+1} + \frac{2}{5}\frac{1}{x^2+1}$, so

$\int \frac{x^2 \; dx}{(x-2)(x^2+1)} = \frac{4}{5}\int \frac{1}{x-2} \; dx + \frac{1}{5}\int \frac{x}{x^2+1} \; dx + \frac{2}{5}\int \frac{1}{x^2+1} \; dx$

Setting $u = x-2$, $du = dx$, so

$\int \frac{1}{x-2} \; dx = \int \frac{1}{u} \; du\Big|_{u=x-2} = \ln|u| + c\Big|_{u=x-2} = \ln|x-2| + c$.

Setting $u = x^2+1$, $du = 2x \; dx$, so $x \; dx = \frac{1}{2}du$, so

$\int \frac{x}{x^2+1} \; dx = \frac{1}{2}\int \frac{1}{u} \; du\Big|_{u=x^2+1} = \frac{1}{2}\ln|u| + c\Big|_{u=x^2+1} = \frac{1}{2}\ln|x^2+1| + c$

$\int \frac{1}{x^2+1} \; dx = \text{Arctan}(x) + c$.

So $\int \frac{x^2 \; dx}{(x-2)(x^2+1)} = \frac{4}{5}\int \frac{1}{x-2} \; dx + \frac{1}{5}\int \frac{x}{x^2+1} \; dx + \frac{2}{5}\int \frac{1}{x^2+1} \; dx$

$= \frac{4}{5}\ln|x-2| + \frac{1}{5}\frac{1}{2}\ln|x^2+1| + \frac{2}{5}\text{Arctan}(x) + c$

$= \frac{4}{5}\ln|x-2| + \frac{1}{10}\ln|x^2+1| + \frac{2}{5}\text{Arctan}(x) + c$.

6. $\int \text{Arcsin}(x) \; dx = (*)$

By parts: $u = \text{Arcsin}(x)$, so $du = \frac{1}{\sqrt{1-x^2}} \; dx$, and $dv = dx$, so $v = x$. Then

$(*) = x\text{Arcsin}(x) - \int \frac{x}{\sqrt{1-x^2}} \; dx$; this integral we can do by substitution:

$w = 1-x^2$, so $dw = -2x \; dx$, so $(*) = x\text{Arcsin}(x) + \frac{1}{2}\int \frac{du}{\sqrt{u}}\Big|_{u=1-x^2}$

$= x\text{Arcsin}(x) + \frac{1}{2}\int u^{1/2} \; du\Big|_{u=1-x^2} = \text{Arcsin}(x) + \frac{1}{2}2u^{1/2} + c\Big|_{u=1-x^2}$

$= \text{Arcsin}(x) + \sqrt{1-x^2} + c$

7. $\int \frac{x^2}{\sqrt{1-x^2}} \; dx = (**)$

By trig substitution: $x = \sin u$, so $dx = \cos u \; du$ and $\sqrt{1-x^2} = \cos u$, so

$(**) = \int \frac{\sin^2 u}{\cos u}\cos u \; du\Big|_{x=\sin u} = \int \sin^2 u \; du\Big|_{x=\sin u}$

$= \int \frac{1}{2}(1 - cos(2u) \; du\Big|_{x=\sin u} = \frac{1}{2}(u - \frac{1}{2}\sin(2u)) + c\Big|_{x=\sin u}$

$= \frac{1}{2}u - \frac{1}{2}\sin u \cos u + c\Big|_{x=\sin u}$

But if $x = \sin u$, then $u = \text{Arcsin}(x)$, and $\cos u = \sqrt{1-x^2}$, so

$(**) = \frac{1}{2}\text{Arcsin}(x) - \frac{1}{2}x\sqrt{1-x^2} + c$

9. $\int_1^3 \frac{x}{(x+1)(x+5)} \; dx \; (***)$

$\dfrac{x}{(x+1)(x+5)} = \dfrac{A}{x+1} + \dfrac{B}{x+5} = \dfrac{A(x+5)+B(x+1)}{(x+1)(x+5)}$, so we need

$x = A(x+5) + B(x+1)$. Setting $x = -5$, we get $-5 = B(-4)$, so $B = \dfrac{5}{4}$.

Setting $x = -1$, we get $-1 = A(4)$, so $A = -\dfrac{1}{4}$.

So (***) $= \displaystyle\int_1^3 -\frac{1}{4}\frac{1}{x+1} + \frac{5}{4}\frac{1}{x+5}\ dx$

$= -\dfrac{1}{4}\ln|x+1| + \dfrac{5}{4}\ln|x+5|\Big|_1^3$ (do u-subs to compute each antiderivative)

$= -\dfrac{1}{4}(\ln(4)-\ln(2)) + \dfrac{5}{4}(\ln(8)-\ln(6)) = \dfrac{5}{4}\ln(\dfrac{4}{3}) - \dfrac{1}{4}\ln(2)$

$= \ln((\dfrac{4}{3})^{5/4}(\dfrac{1}{2})^{1/4})$