10. Does the integral $\int_1^\infty \frac{1}{e^x - x}\,dx$ converge or diverge?

(Note: 'Yes' is not considered a correct answer.....)

$\frac{1}{e^x-x}$ looks "like" $\frac{1}{e^x}$ (or $\frac{1}{-x}$ or...)

$$\int_1^\infty \frac{1}{e^x}\,dx = \int_1^\infty e^{-x}\,dx = -e^{-x}\Big|_1^\infty$$

$$= \lim_{b\to\infty}\left(-e^{-x}\Big|_1^b\right) = \lim_{b\to\infty}\left(-e^{-b}-(-e^{-1})\right) = \lim_{b\to\infty} \frac{1}{e} - \underbrace{\frac{1}{e^b}}_{\to 0}$$

$$= \frac{1}{e} < \infty.$$

$$\lim_{x\to\infty} \frac{\frac{1}{e^x}}{\frac{1}{e^x-x}} = \lim_{x\to\infty} \frac{e^x-x}{e^x} = \lim_{x\to\infty} 1 - \frac{x}{e^x} = 1 - \lim_{x\to\infty} \frac{x}{e^x}$$

$$= 1 - \lim_{x\to\infty} \frac{1}{e^x} = 1 - 0 = 1 \neq 0, \infty$$

L'Hopital!

So since $\int_1^\infty \frac{1}{e^x}\,dx$ converges,

$\int_1^\infty \frac{1}{e^x-x}\,dx$ converges by <u>limit</u> <u>comparison</u>

Soltions

Name:

Math 1720 Section 022

Exam number 3

Show all work. How you get your answer is just as important, if not more important, than the answer itself. If you think it, write it!

In problems 1, 2, and 3, find the indicated limits. (10 pts. each)

1. $\lim_{n\to\infty} \frac{1-4n^3+4n}{3n^2-n+3} = \infty$

$$\frac{1-4n^3+4n}{3n^2-n+3} = \frac{\frac{1}{n^2}-4n+\frac{4}{n}}{3-\frac{1}{n}+\frac{3}{n^2}} \longrightarrow \underline{\underline{\infty}}$$

$\frac{1}{n^2} \to 0$, $4n \to \infty$, $\frac{4}{n} \to 0$, $\frac{1}{n} \to 0$, $\frac{3}{n^2} \to 0$

2. $\lim_{n\to\infty} \frac{n^{n+\frac{1}{n}}}{(n+2)^n} = \lim \left(\frac{n}{n+2}\right)^n n^{1/n}$

$$= \frac{n^{1/n}}{\left(1+\frac{2}{n}\right)^n} \longrightarrow \frac{1}{e^2}$$

a_n

3. $\lim_{n\to\infty} \frac{n^2 + 4n\cos n - 1}{2n^2 + 15} =$ $\lim \dfrac{1 + \frac{\cos n}{n} - \frac{1}{n^2}}{2 + \frac{15}{n^2}}$

$$\frac{1 - \frac{1}{n} - \frac{1}{n^2}}{2 + \frac{15}{n^2}} \le \frac{1 + \frac{\cos n}{n} - \frac{1}{n^2}}{2 + \frac{15}{n^2}} \le \frac{1 + \frac{1}{n} - \frac{1}{n^2}}{2 + \frac{15}{n^2}}$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$\frac{1}{2} \qquad \lim a_n = \frac{1}{2} \qquad \frac{1}{2}$$

by Sandwich Thm

4. (10 pts.) Find the indicated sum: $\sum_{n=0}^{\infty} \frac{2^n + 1 + 3^n}{5^n} =$

$$= \sum_{n=0}^{\infty} \frac{2^n}{5^n} + \sum_{n=0}^{\infty} \frac{1}{5^n} + \sum_{n=0}^{\infty} \frac{3^n}{5^n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{2}{5}\right)^n + \sum_{n=0}^{\infty} \left(\frac{1}{5}\right)^n + \sum_{n=0}^{\infty} \left(\frac{3}{5}\right)^n$$

$$= \frac{1}{1 - \frac{2}{5}} + \frac{1}{1 - \frac{1}{5}} + \frac{1}{1 - \frac{3}{5}} = \frac{5}{3} + \frac{5}{4} + \frac{5}{2}$$

$$= \frac{20 + 15 + 30}{12} = \frac{65}{12}$$

Do any SIX (6) of the following EIGHT (8) problems (10 pts. each)

Using any (legal) method (other than psychic powers), determine the convergence or divergence of the following infinite series (be sure to show sufficient work so that the method used in determining conv/div can be understood):

5. $\sum_{n=0}^{\infty} \frac{n-3}{n+1}$

$$\frac{n-3}{n+1} = \frac{1-\frac{3}{n}}{1+\frac{1}{n}} \to \frac{1}{1} \neq 0$$

so series diverges.
(nth term test)

6. $\sum_{n=1}^{\infty} \frac{n^2+n}{n!} = \sum a_n$

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^2+(n+1)}{\cancel{(n+1)!}} \cdot \frac{\cancel{n!}}{n^2+n}$$
$(n+1)$

$$= \frac{n+1+1}{n^2+n} = \frac{n+2}{n^2+n} = \frac{1+\frac{2}{n}}{n+1} \to 0 < 1$$

so series converges
(Ratio Test)

7. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^{2/3}}$ $\qquad f(x) = \dfrac{1}{x(\ln x)^{2/3}}$

$$\int_2^{\infty} \frac{dx}{x(\ln x)^{2/3}} = \begin{pmatrix} u = \ln x \\ du = \frac{1}{x}dx \end{pmatrix} = \int_{\ln 2}^{\infty} \frac{du}{u^{2/3}} = 3u^{1/3}\Big|_{\ln 2}^{\infty}$$

as $u \to \infty$ ~~the~~ $u^{1/3} \to \infty$ so integral diverges

so series diverges (Integral Test)

8. $\displaystyle\sum_{n=2}^{\infty} \frac{n^2}{(\ln n)^n} = \sum a_n$

$$a_n^{1/n} = \left(\frac{n^2}{(\ln n)^n}\right)^{1/n} = \frac{(n^2)^{1/n}}{\ln n} = \frac{\overbrace{(n^{1/n})^2}^{1}}{\underbrace{(\ln n)}_{\infty}} \to 0 < 1$$

so series converges (Root Test)

9. $\sum_{n=2}^{\infty} \frac{n}{\sqrt{n^9+3n-4}}$ $= \sum c_n$ $b_n = \frac{n}{\sqrt{n^9}} = \frac{1}{n^{7/2}}$

$$\frac{c_n}{b_n} = \frac{\cancel{n}}{\sqrt{n^9+3n-4}} \cdot \frac{\sqrt{n^9}}{\cancel{n}} = \sqrt{\frac{n^9}{n^9+3n-4}} = \sqrt{\frac{1}{1-\frac{3}{n^8}-\frac{4}{n^9}}}$$

$\rightarrow 1 \neq 0, \infty$ so since

$\sum \frac{1}{n^{7/2}}$ conv (p-series, $p = 7/2 > 1$)

$\sum c_n$ conv. Limit comparison Test

10. $\sum_{n=0}^{\infty} ne^{-n^2}$ $f(x) = xe^{(-x^2)}$

$$\int_0^{\infty} xe^{(-x^2)}dx = \qquad u = -x^2, \quad du = -2x\,dx$$

$$= \int_0^{-\infty} -\tfrac{1}{2}e^u du = -\tfrac{1}{2}e^u \Big|_0^{-\infty} \qquad e^u \to 0 \text{ as } u \to -\infty$$

$$= 0 - \left(-\tfrac{1}{2}e^0\right) = \tfrac{1}{2} < \infty$$

so series converges (Integral Test)

11. $\sum_{n=3}^{\infty} \frac{n}{n^2-6} = \sum a_n$ $\qquad b_n = \frac{1}{n}$

$$\frac{a_n}{b_n} = \frac{n}{n^2-6} \cdot \frac{n}{1} = \frac{n^2}{n^2-6} = \frac{1}{1-\frac{6}{n^2}} \to 1 \neq 0, \infty$$

so since $\sum \frac{1}{n}$ diverges, so does $\sum \frac{n}{n^2-6}$

By Limit Comparison Test.

12. $\sum_{n=0}^{\infty} \frac{2n^3-1}{5^n} = \sum a_n$

$$\frac{a_{n+1}}{a_n} = \frac{2(n+1)^3-1}{5^{n+1}} \cdot \frac{5^n}{2n^3-1}$$

$$= \frac{2(n+1)^3-1}{2n^3-1} \cdot \frac{1}{5} = \frac{2n^3+6n^2+6n+1}{2n^3-1} \cdot \frac{1}{5}$$

$$= \frac{2+\frac{6}{n}+\frac{6}{n^2}+\frac{1}{n^3}}{2-\frac{1}{n^3}} \cdot \frac{1}{5} \to \frac{2}{2} \cdot \frac{1}{5} = \frac{1}{5} < 1$$

so series converges by Ratio Test

4-3. $\sum_{n=2}^{\infty} \frac{n-1}{6n+1}$

$$\frac{n-1}{6n+1} = \frac{1-\frac{1}{n}}{6+\frac{1}{n}} \to \frac{1}{6} \neq 0$$

∴ series diverges.

5. Find the radius of convergence of the following power series (5 pts. each):

(a): $\sum_{n=0}^{\infty} \frac{n5^n}{n!} x^n = \sum a_n x^n \qquad a_n = \frac{n5^n}{n!}$

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)5^{n+1}}{(n+1)!} \cdot \frac{n!}{n5^n} = \frac{5}{n} \to 0 = L$$

$\frac{1}{L} = \infty = R$ = rad. of conv.

(b): $\sum_{n=1}^{\infty} \frac{n}{3^n - 1}(x-2)^n = \sum a_n (x-2)^n \qquad a_n = \frac{n}{3^n-1}$

$$\frac{a_{n+1}}{a_n} = \frac{n+1}{3^{n+1}-1} \cdot \frac{3^n-1}{n} = \left(\frac{n+1}{n}\right) \cdot \frac{3^n-1}{3^{n+1}-1}$$

$$= \left(\frac{n+1}{n}\right) \cdot \frac{1-\frac{1}{3^n}}{3-\frac{1}{3^n}} \to \frac{1}{3} = L \qquad R = \frac{1}{L} = 3 = \text{rad of conv.}$$

6. (15 points) Find the Taylor polynomial, of degree 3, centered at a=0, for the function

$$f(x) = (x+3)^{5/2}$$

$f(0) = 3^{5/2}$

$f'(0) = \frac{5}{2} 3^{3/2}$

$f''(0) = \frac{15}{4} 3^{1/2}$

$f'''(0) = \frac{15}{8} 3^{-1/2}$

$f'(x) = \frac{5}{2}(x+3)^{3/2}$

$f''(x) = \frac{15}{4}(x+3)^{1/2}$

$f'''(x) = \frac{15}{8}(x+3)^{-1/2}$

$$P_3(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3$$

$$= 3^{5/2} + \frac{5}{2}3^{3/2}x + \frac{15}{4}\frac{3^{1/2}}{2!}x^2 + \frac{15}{8}\frac{3^{-1/2}}{3!}x^3$$

$$3^{5/2} + \frac{5}{2}3^{3/2}x + \frac{15}{8}3^{1/2}x^2 + \frac{15}{48}3^{-1/2}x^3$$

$\frac{5}{16}$

7. (10 pts.) Find the power series (centered at 0) for the function

$$f(x) = \frac{1}{(1+x^2)^2}$$

(Hint: Start with a series for $\frac{1}{1-x}$, and build from there... (derivative? integral?))

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$\frac{1}{1+x^2} = \sum \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n$$

$$= \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

$$\frac{d}{dx}\left(\frac{1}{1+x^2}\right) = \frac{-2x}{(1+x^2)^2} = \sum_{n=0}^{\infty} (-1)^n 2n x^{2n-1}$$

$$\frac{1}{(1+x^2)^2} = \frac{-1}{\cancel{2}x} \sum_{n=0}^{\infty} (-1)^n \cancel{2}n x^{2n-1}$$

$$= \boxed{\sum_{n=0}^{\infty} (-1)^{n+1} n x^{2n-2}}$$

or:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \ ; \quad \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{-1}{(1-x)^2} = \sum_{n=0}^{\infty} n x^{n-1}$$

$$\frac{1}{(1-x)^2} = -\sum_{n=0}^{\infty} n x^{n-1}$$

$$\frac{1}{(1+x^2)^2} = \frac{1}{(1-(-x^2))^2} = -\sum_{n=0}^{\infty} n (-x^2)^{n-1}$$

$$\boxed{= \sum_{n=0}^{\infty} n (-1)^{n+1} x^{2n-2}}$$