Math 107H

Topics for the first exam

Chapter 4: Integration

Antiderivatives. Integral calculus is all about finding areas of things, e.g. the area between the graph of a function f and the x-axis. This will, in the end, involve finding a function F whose *derivative* is f.

F is an *antiderivative* (or (indefinite) *integral*) of f if $F'(x) = f(x)$.
Notation: $F(x) = \int f(x)\ \mathrm{d}x$; it <u>means</u> $F'(x) = f(x)$; "the integral of f of x dee x"

Basic list:

$\int x^n\ \mathrm{d}x = \dfrac{x^{n+1}}{n+1} + C$ (provided $n \neq -1$) $\qquad \int 1/x\ \mathrm{d}x = \ln|x| + C$

$\int \sin(kx)\ \mathrm{d}x = \dfrac{-\cos(kx}{k} + C \qquad \int \cos(kx)\ \mathrm{d}x = \dfrac{\sin(kx)}{k} + C$

$\int \sec^2 x\ \mathrm{d}x = \tan x + C \qquad \int \csc^2 x\ \mathrm{d}x = -\cot x + C$

$\int \sec x \tan x\ \mathrm{d}x = \sec x + C \qquad \int \csc x \cot x\ \mathrm{d}x = -\csc x + C$

$\int e^x\ \mathrm{d}x = e^x + C$

$\int \tan x\ \mathrm{d}x = \ln|\sec x| + C \qquad \int \sec x\ \mathrm{d}x = \ln|\sec x + \tan x| + C$

$\int \cot x\ \mathrm{d}x = \ln|\sin x| + C \qquad \int \csc x\ \mathrm{d}x = -\ln|\csc x + \cot x| + C$

Most differentiation rules can be turned into integration rules (although some are harder than others; some we will wait awhile to discover.)
Basic integration rules: sum and constant multiple rules are straighforward to reverse: for k=constant,

$\int k \cdot f(x)\ \mathrm{d}x = k \int f(x)\ \mathrm{d}x \qquad \int (f(x) \pm g(x)\ \mathrm{d}x = \int f(x)\ \mathrm{d}x \pm \int g(x)\ \mathrm{d}x$

Sums and Sigma Notation. Idea: a lot of things can estimated by adding up alot of tiny pieces.

Sigma notation: $\displaystyle\sum_{i=1}^{n} a_i = a_1 + \cdots a_n$; just add the numbers up

Formal properties: $\displaystyle\sum_{i=1}^{n} k a_i = k \sum_{i=1}^{n} a_i \qquad \sum_{i=1}^{n} (a_i \pm b_i) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i$

Some things worth adding up:
length of a curve: approximate curve by a collection of straight line segments

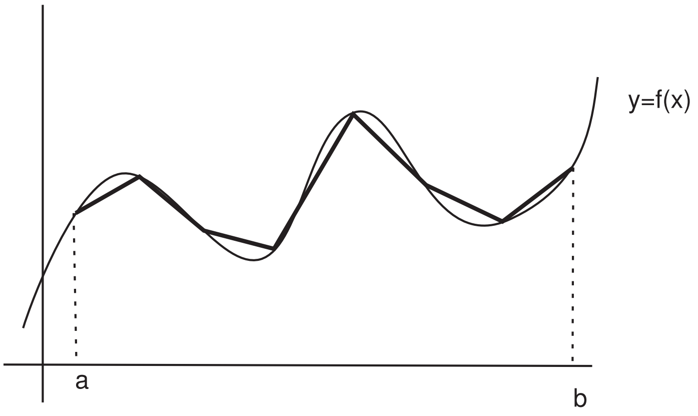
length of curve $\approx \sum$(length of line segments)

distance travelled = (average velocity)(time of travel)
over short periods of time, avg. vel. $\approx$ instantaneous vel.
so distance travelled $\approx \sum$(inst. vel.)(short time intervals)

Average value of a function

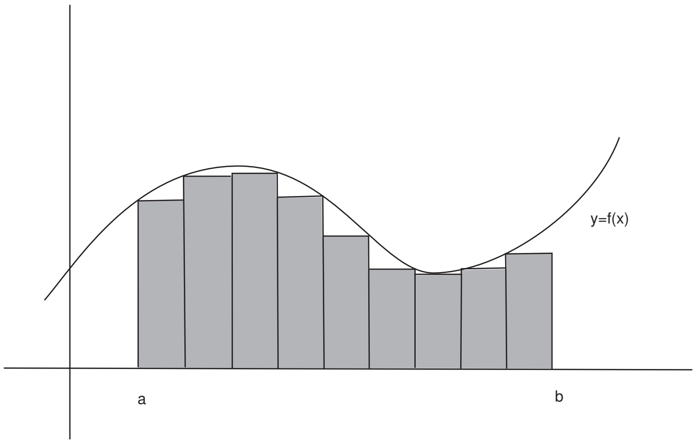
Average of n numbers: add the numbers, divide by n . For a function, add up lots of values of f, divide by number of values.

avg. value of $f \approx \dfrac{1}{n} \displaystyle\sum_{i=1}^{n} f(c_i)$



Area and Definite Integrals. Probably the most important thing to approximate by sums: area under a curve.

Idea: approximate region b/w curve and x-axis by things whose areas we can easily calculate: **rectangles**!



Area between graph and x-axis $\approx \sum$ (areas of the rectangles) $= \sum_{i=1}^{n} f(c_i)\Delta x_i$

We <u>define</u> the area to be the <u>limit</u> of these sums as the number of rectangles goes to ∞ (i.e., the width of the rectangles goes to 0), and call this the *definite integral* of f from a to b:

$$\int_a^b f(x)\ dx \ = \ \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x_i$$

When do such limits exist?

Theorem If f is continuous on the interval $[a, b]$, then $\int_a^b f(x)\ dx$ exists.

(i.e., the area under the graph <u>is</u> approximated by rectangles.)

Properties of definite integrals

First note: the sum used to define a definite integral doesn't <u>need</u> to have $f(x) \geq 0$; the limit still makes sense. When f <u>is</u> bigger than 0, we <u>interpret</u> the integral as area under the graph.

Basic properties of definite integrals:

$\int_a^a f(x)\ dx = 0$ $\qquad$ $\int_b^a f(x)\ dx = -\int_a^b f(x)\ dx$

$\int_a^b kf(x)\ dx = k\int_a^b f(x)\ dx$ $\qquad$ $\int_a^b f(x) \pm g(x)\ dx = \int_a^b f(x)\ dx \pm \int_a^b g(x)\ dx$

$\int_a^b f(x)\ dx + \int_b^c f(x)\ dx = \int_a^c f(x)\ dx$

If $m \leq f(x) \leq M$ for all x in $[a,b]$, then $m(b-a) \leq \int_a^b f(x)\ dx \leq M(b-a)$

More generally, if $f(x) \leq g(x)$ for all x in $[a,b]$, then $\int_a^b f(x)\ dx \leq \int_a^b g(x)\ dx$

Average value of f : formalize our old idea! $\qquad \text{avg}(f) = \dfrac{1}{b-a}\int_a^b f(x)\ dx$

Mean Value Theorem for integrals: If f is continuous in $[a,b]$, then there is a c in $[a,b]$ so that $f(c) = \dfrac{1}{b-a}\int_a^b f(x)\ dx$

The fundamental theorems of calculus. Formally, $\int_a^b f(x)\ dx$ depends on a and b. Make this explicit:

$\int_a^x f(t)\ dt = F(x)$ is a function of x.

$F(x) =$ the area under the graph of f, from a to x.

Fund. Thm. of Calc (# 2): If f is continuous, then $F'(x) = f(x)$ $\quad$ (F is an antiderivative of f !)

Since any two antiderivatives differ by a constant, and $F(b) = \int_a^b f(t)\ dt$, we get

Fund. Thm. of Calc (# 1): If f is continuous, and F is an antiderivative of f, then $\int_a^b f(x)\ dx = F(b) - F(a) = F(x)\ |_a^b$

Ex: $\int_0^\pi \sin x\ dx = (-\cos\pi) - (-\cos 0) = 2$

Building antiderivatives:

$F(x) = \int_a^x \sqrt{\sin t}\ dt$ is an antiderivative of $f(x) = \sqrt{\sin x}$

$G(x) = \int_{x^2}^{x^3} \sqrt{1+t^2}\ dt = F(x^3) - F(x^2)$, where

$F'(x) = \sqrt{1+x^2}$, so $G'(x) = F'(x^3)(3x^2) - F'(x^2)(2x)$...

Integration by substiution. The idea: reverse the chain rule!

If $g(x) = u$, then $\dfrac{d}{dx}f(g(x)) = \dfrac{d}{dx}f(u) = f'(u)\ \dfrac{du}{dx}$

so $\int f'(u)\ \dfrac{du}{dx}\ dx = \int f'(u)\ du = f(u) + c$

$\int f(g(x))g'(x)\ dx$; set $u = g(x)$, then $du = g'(x)\ dx$, so $\int f(g(x))g'(x)\ dx = \int f(u)\ du$, <u>where</u> $u = g(x)$

Example: $\int x(x+2-3)^4\ dx$; set $u = x^2 - 3$, so du=$2x\ dx$. Then

$$\int x(x+2-3)^4\ dx = \frac{1}{2}\int (x+2-3)^4 2x\ dx = \frac{1}{2}\int u^4\ du\ |_{u=x^2-3} =$$

$$\frac{1}{2}\frac{u^5}{5} + c\ |_{u=x^2-3} = \frac{(x^2-3)^5}{10} + c$$

The three most important points:

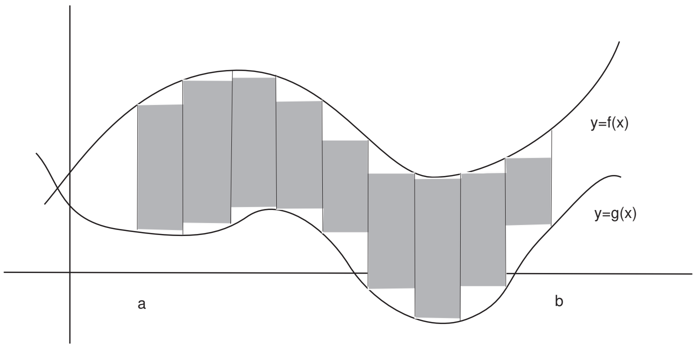
1. Make sure that you calculate (and then set aside) your du before doing step 2!
2. Make sure everything gets changed from x's to u's
3. **<u>Don't</u>** push x's through the integral sign! They're <u>not</u> constants!

We can use u-substitution directly with a definite integral, provided we remember that $\displaystyle\int_a^b f(x)\ dx$ <u>really</u> means $\displaystyle\int_{x=a}^{x=b} f(x)\ dx$, and we remember to change <u>all</u> of the x's to u's!

Ex: $\displaystyle\int_1^2 x(1+x^2)^6\ dx$; set $u = 1+x^2$, $du = 2x\ dx$. when $x = 1$, $u = 2$; when $x = 2$, $u = 5$; so $\displaystyle\int_1^2 x(1+x^2)^6\ dx = \frac{1}{2}\int_2^5 u^6\ du = ...$

Chapter 5: Applications of integration

Area between curves. Region between two curves; approximate by rectangles



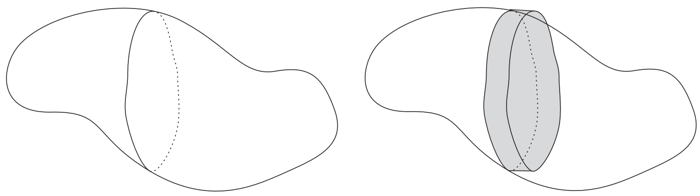
$$\text{Area} = \int_{left}^{right} (top) - (bottom)\ dx = \int_a^b f(x) - g(x)\ dx$$

Integrate dy: Area $= \displaystyle\int_{bottom}^{top} (right) - (left)\ dy$

If what the function at top/bottom <u>is</u> changes, cut the interval into pieces, and use $\displaystyle\int_a^b = \int_a^c + \int_c^b$

Sometimes to calculate area between $f(x)$ and $g(x)$, need to first figure out limits of integration; solve $f(x) = g(x)$, then decide whicxh one is bigger in between each pair of solutions.

Volume by slicing. To calculate volume, aprroximate region by objects whose volume we can calculate.

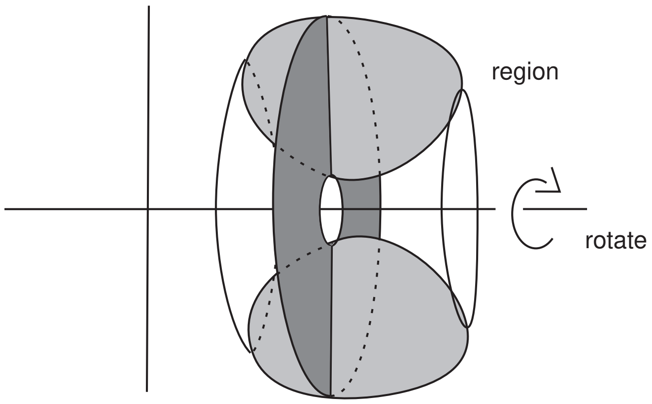


Volume $\approx \sum$(volumes of 'cylinders')

$$= \sum(\text{area of base})(\text{height}) = \sum(\text{area of cross-section})\Delta x_i \ .$$

So volume $= \displaystyle\int_{left}^{right}$ (area of cross section) dx

Solids of revolution: disks and washers. Solid of revolution: take a region in the plane and revolve it around an axis in the plane.



take cross-sections perpendicular to axis of revolution ; cross-section = disk (area=πr^2) or washer (area=$\pi R^2 - \pi r^2$)

rotate around x-axis: write r (and R) as functions of x, integrate dx

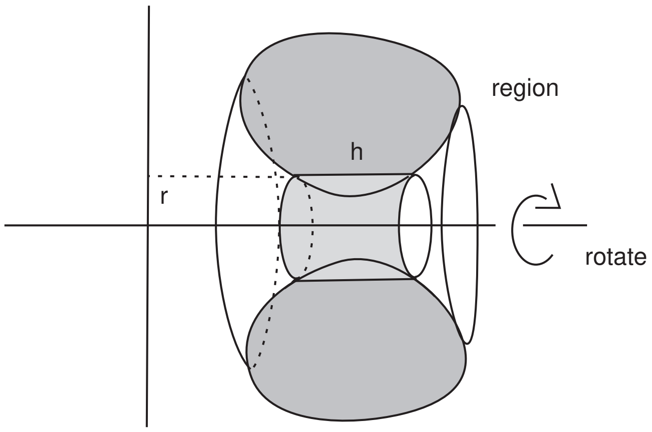
rotate around y-axis: write r (and R) as functions of y, integrate dy

Otherwise, everything is as before: volume $= \displaystyle\int_{left}^{right} A(x)\,dx$ or volume $= \displaystyle\int_{bottom}^{top} A(y)\,dy$

The same is true if axis is parallel to $x-$ or $y-$axis; r and R just change (we add a constant).

Cylindrical shells. Different picture, same volume! Solid of revolution; use cylinders centered on the axis of revolution. The intersection is a cylinder, with area = (circumference)(height) $= 2\pi rh$

volume $= \displaystyle\int_{left}^{right}$ (area of cylinder) dx or $\displaystyle\int_{bottom}^{top}$ (area of cylinder) dy!)



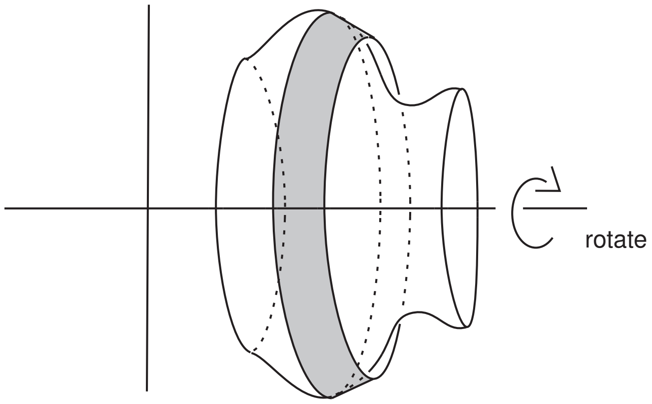
revolve around vertical line: integrate dx

revolve around horizontal line: integrate dy

Ex: region in plane between $y = 4x$, $y = x^2$, revolved around y-axis

left=0, right=4, $r = x$, $h = (4x - x^2)$ volume $= \displaystyle\int_0^4 2\pi x(4x - x^2)\ dx$

Arclength and surface area



Arclength. Idea: approximate a curve by lots of short line segments; length of curve ≈ sum of lengths of line segments.

Line segment between $(c_i, f(c_i))$ and $(c_{i+1}, f(c_{i+1}))$:

$$\sqrt{1 + (\frac{f(c_{i+1}) - f(c_i)}{c_{i+1} - c_i})^2} \cdot (c_{i+1} - c_i) \approx \sqrt{1 + (f'(c_i))^2} \cdot \Delta x_i$$

<u>So</u> length of curve $= \displaystyle\int_{left}^{right} \sqrt{1 + (f'(x))^2}\ dx$

The problem: integrating $\sqrt{1 + (f'(x))^2}$! Sometimes, $1 + (f'(x))^2$ turns out to be a perfect square.....

Surface area. Idea: find the area of a surface (of revolution) by approximating the surface by things whose area we can figure out. Frustrum of a cone!

area of frustrum $= \pi \cdot \big(f(c_{i+1}) + f(c_i)\big) \cdot \sqrt{1 + \Big(\frac{f(c_{i+1}) - f(c_i)}{c_{i+1} - c_i}\Big)^2} \cdot (c_{i+1} - c_i)$

$\approx 2\pi f(c_i \cdot \sqrt{1+(f'(c_i))^2} \cdot \Delta x_i$. So area of surface = $\displaystyle\int_{left}^{right} 2\pi f(x)\sqrt{1+(f'(x))^2}\ dx$

The problem: same problem! But sometimes it's possible to do.... Ex: for $f(x) = \sqrt{r^2 - x^2}$, the thing to integrate simplifies to: $2\pi r$!

Work. In physics, one studies the behavior of objects when acted upon by various *forces*. Newton's Laws provide the basic connection between a force acting on an object and the effect it has on its motion:

$$F = ma\ ; \qquad \text{Force} = \text{mass} \times \text{acceleration}$$

Two basic quantites to compute, when you know the force, are *impulse* and *work*.

Impulse measures the effect of a force over time. If a constant force F is applied to an object, over a time interval of length T, then the impulse imparted to the object is Impulse $= J = F \cdot T$. But typically the force being apllied will not be constant. Then we do what we usually do: look at the impulse generated by the force over a short time interval (where ther force is effectively constant), and add up the impulses imparted over all of these little intervals.

$J \approx \sum F(t_i)\ \Delta t$, which looks suspiciously like an integral. So we define $J = \int_0^T F(t)\ dt$

But in classical physics, where $F(t) = m \cdot a(t) = m \cdot x''(t)$, if we can treat m as a constant, then we can integrate F, so

$$J = m \cdot x'(T) - m \cdot x'(0) = m \cdot v(T) - m \cdot v(0)$$

is the change of momentum of the object.

In physics, *work* represents force being applied across a distance. If a constant force F is applied to an object, which moves the object a distance D, then the work done on the object is $W = F \cdot D$. Again, if the force applied across this distance is not constant, then we interpret work, in stead, as an integral, by cutting the distance covered into small pieces of length δx :

$$W \approx \sum F(x_i)\ \Delta x\ , \text{ so } W = \int_0^D F(x)\ dx$$

An interesting application of these ideas comes when trying to compute the amount of work necessary to pump out a tank of some known shape. If the tank has height D (we will think of the top of the tank as being at $x = 0$ and the bottom being at $x = D$), and at height X our cross-section of the tank has area $A(x)$, then if (as when we computed volume) we think of the fluid in the tank as being a stack of cylinders with height Δx, the work necessary to lift the slide at height x to the top of the tank will be

$$W = (\text{force})(\text{distance}) = (m \cdot g) \cdot x = ((A(x) \cdot \Delta x)\rho g) \cdot x$$

where ρ is the density of the fluid, m = mass = (volume)(density), and g is the accelration due to gravity (which is the force we need to overcome to push the fluid up out of the tank). Therefore, the work done to empty the tank is approximated by a sum of such quantities, which in turn models a definite integral; the work done in emptying the tank is

$$W = \rho g \int_0^D xA(x)\ dx$$