Cyclic Graphs, Their Edge Ideals, and a Comparison of Powers

Thomas Kamp and Jason Vander Woude

Dordt College

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Math on the Northern Plains

Background and Relevance

Graph Theory and Commutative Algebra are closely related. By studying the connections between the two subjects, we can gain a better understanding of both of them.

- The Waldschmidt constant for edge ideals is directly related to the chromatic and clique numbers for graphs.

\[ \frac{\chi(G)}{\chi(G) - 1} \leq \alpha(I(G)) \leq \frac{\omega(G)}{\omega(G) - 1} \]

- If the complement of a graph is chordal, then we are able to find a nice characterization of the free resolution of the $t$-th power of the edge ideal.
Graphs:

- A collection of vertices and edges connecting those vertices

Cycles:
A Vertex Cover is a set of vertices for a graph where every edge is connected to at least one vertex in the set.
Monomials

Polynomials
\[ x^2 + 5x + 6 \quad 2x^4 + 7x^2 + 3x + 1 \]

Multivariable Polynomials
\[ x_1^2x_2x_3^3 + 5x_1x_4 + 6x_2 \quad 2x_1^4 + 7x_1^2x_4x_5^3x_6^7 + 3x_1^5 + 1 \]

Monomials
\[ x_1^2x_2x_3^3 \quad x_1^2x_4x_5^3x_6^7 \]
Ideals

An ideal is a set with special additive and multiplicative properties.

Example:

- The set of all even numbers (all numbers divisible by 2) is an ideal of the integers, and we’ll call this ideal $E$.
  - The sum of any two even numbers is an even number.
  - Any integer times an even number is an even number.
- $E^2$ is the set of all numbers divisible by $2^2 = 4$.
- $E^3$ is the set of all numbers divisible by $2^3 = 8$.

An Edge Ideal is the set of all monomials that are divisible by at least one edge (or a pair of adjacent vertices).
Definitions and Research Goals

\[ I = (\{x_ix_j \mid \{x_i, x_j\} \in V(G)\}) \]
\[ I^t = (\{x_1x_1x_2x_2 \cdots x_itx_jt \mid \{x_i, x_j\} \in V(G)\}) \]

\[ w_{V'}(x^a) = \sum_{x_i \in V'} a_i \]

\[ I^{(t)} = (\{x^a \mid \text{for all minimal vertex covers } V', w_{V'}(x^a) \geq t\}) \]
\[ L(t) = \{x^a \mid \deg(x^a) \geq 2t \text{ and } \forall V', w_{V'}(x^a) \geq t\} \]
\[ D(t) = \{x^a \mid \deg(x^a) < 2t \text{ and } \forall V', w_{V'}(x^a) \geq t\} \]

\[ I^{(t)} = (L(t)) + (D(t)) \]

Goal: \[ I^t = (L(t)) \]
\[ \bullet I^t \subseteq (L(t)) \quad \text{[trivial]} \]
\[ \bullet I^t \supseteq (L(t)) \quad \text{[difficult]} \]
Graphing Monomials

We graphically represent the monomials by reducing each pair of consecutive vertices to an edge so that we get *as many edges as we can*.

Consider the monomial $x_1^2 x_2^3 x_3 x_4 x_5^2$: 

```plaintext
\begin{align*}
x_1 & \quad x_1 \\
x_2 & \quad x_2 \quad x_2 \\
x_3 & \\
x_4 & \\
x_5 & 
\end{align*}
```

![Diagram of a graph with vertices $x_1, x_2, x_3, x_4, x_5$ connected by edges.](image)
Graphing Monomials

We graphically represent the monomials by reducing each pair of consecutive vertices to an edge so that we get \textit{as many edges as we can}.

Consider the monomial $x_1^2x_2^3x_3x_4x_5^2$:

\[
x_1^2x_2^3x_3x_4x_5^2 = (x_1x_2)(x_3x_4)(x_1x_5)x_2^2
\]
Graphing Monomials

We graphically represent the monomials by reducing each pair of consecutive vertices to an edge so that we get \textit{as many edges as we can}.

Consider the monomial $x_1^2x_2^3x_3x_4x_5^2$:

$$x_1^2x_2^3x_3x_4x_5^2 = (x_1x_2)^2(x_2x_3)(x_4x_5)$$
Graphing Monomials

When a monomial cannot reduce perfectly to a product of edges, it creates *ancillary* vertices.

Consider the monomial $x_1^2 x_2^3 x_3 x_4 x_5^2$:

$$x_1^2 x_2^3 x_3 x_4 x_5^2 = (x_1 x_2)^2 (x_2 x_3) (x_4 x_5) x_5$$
Graphing Monomials

Each monomial does not necessarily correspond to a unique graph.

Consider $x_1^2 x_2^3 x_3^2 x_4 x_5$

- $(x_1 x_2)^2 (x_2 x_3) x_3 (x_4 x_5)$
- $(x_1 x_2) (x_2 x_3)^2 x_4 (x_5 x_1)$
Graphing Monomials

We had to be concerned with an infinite number of possible graphs of varying sizes, so we wanted to understand what graphs can and cannot look like.

Assuming one of each ancillary.
Graphing Monomials

But even after turning those two ancillaries into an edge, it is still not good enough!

Assuming one of each ancillary.
Graphing Monomials
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Example:

Let $t = 14$ and let $m \notin I^t$. Our goal is to demonstrate that $m \notin (L(t))$.

$$m = x_1^2 x_2^{100} x_3^3 x_4^{20} x_5^2 x_6^3 x_7^{54} x_8^3 x_9^3$$
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ex: \( t = 14 \)
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ex: \( t = 14 \)

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\( (L(t)) \subseteq I^t \)
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\( (L(t)) \subseteq I^t \)

\( (L(t)) = I^t \)
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ex: \( t = 14 \)
\( m \notin I^t \)
\( w_V(m) = \text{num. edges} < t \)
\( m \notin L(t) \)
\( m \notin (L(t)) \)
\( (L(t)) \subseteq I^t \)
\( (L(t)) = I^t \)
\( I^{(t)} = I^t + (D(t)) \)
Theorem (Kamp-Vander Woude)

Let $G$ be an odd cycle of size $2n + 1$, $I$ be its edge ideal, and $V'$ denote a minimal vertex cover. Then

$$I^{(t)} = I^t + \left( \{ x^\alpha | \deg(x^\alpha) < 2t \text{ and } \forall V' \subseteq V(G), w_{V'}(x^\alpha) \geq t \} \right)$$