Cyclic Graphs, Their Edge Ideals, and a Comparison of Powers

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Background and Relevance

- Graph Theory and Commutative Algebra are closely related. By studying the connections between the two subjects, we can gain a better understanding of both of them.
  - The Waldschmidt constant for edge ideals is directly related to the chromatic and clique numbers for graphs.

$$\frac{\chi(G)}{\chi(G)-1} \leq \widehat{\alpha}(I(G)) \leq \frac{\omega(G)}{\omega(G)-1}$$

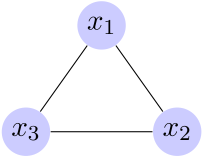
  - If the complement of a graph is chordal, then we are able to find a nice characterization of the free resolution of the t-th power of the edge ideal.

An Introduction to Abstract Algebra and Graph Theory

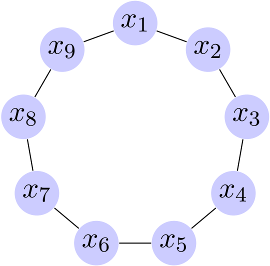
Graphs:

- A collection of vertices and edges connecting those vertices

Cycles:



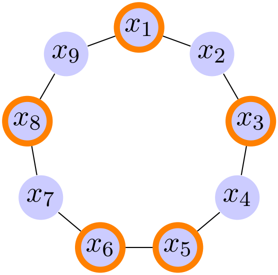
3-Cycle

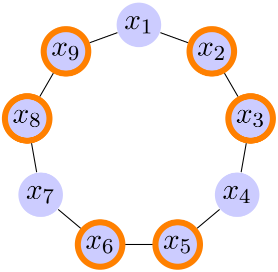


9-Cycle

An Introduction to Abstract Algebra and Graph Theory

A *Vertex Cover* is a set of vertices for a graph where every edge is connected to at least one vertex in the set.





Monomials

Polynomials

$$x^2 + 5x + 6 \qquad 2x^4 + 7x^2 + 3x + 1$$

Multivariable Polynomials

$$x_1^2 x_2 x_3^3 + 5x_1 x_4 + 6x_2 \qquad 2x_1^4 + 7x_1^2 x_4 x_5^3 x_6^7 + 3x_1^5 + 1$$

Monomials

$$x_1^2 x_2 x_3^3 \qquad x_1^2 x_4 x_5^3 x_6^7$$

Ideals

An ideal is a set with special additive and multiplicative properties.

Example:

- The set of all even numbers (all numbers divisible by 2) is an ideal of the integers, and we'll call this ideal E.
  - The sum of any two even numbers is an even number.
  - Any integer times an even number is an even number.
- E^2 is the set of all numbers divisible by $2^2 = 4$.
- E^3 is the set of all numbers divisible by $2^3 = 8$.

An *Edge Ideal* is the set of all monomials that are divisible by at least one edge (or a pair of adjacent vertices).

Definitions and Research Goals

$$I = (\{x_i x_j \mid \{x_i, x_j\} \in V(G)\})$$
$$I^t = (\{x_{i_1}x_{j_1}x_{i_2}x_{j_2}\cdots x_{i_t}x_{j_t} \mid \{x_{i_k}, x_{j_k}\} \in V(G)\})$$

$$w_{V'}(x^{\underline{a}}) = \sum_{x_i \in V'} a_i$$

$$I^{(t)} = (\{x^{\underline{a}} \mid \text{ for all minimal vertex covers } V',\, w_{V'}(x^{\underline{a}}) \geq t\})$$
$$L(t) = \{x^{\underline{a}} \mid \deg(x^{\underline{a}}) \geq 2t \text{ and } \forall V',\, w_{V'}(x^{\underline{a}}) \geq t\}$$
$$D(t) = \{x^{\underline{a}} \mid \deg(x^{\underline{a}}) < 2t \text{ and } \forall V',\, w_{V'}(x^{\underline{a}}) \geq t\}$$

$$I^{(t)} = (L(t)) + (D(t))$$

Goal: $I^t = (L(t))$

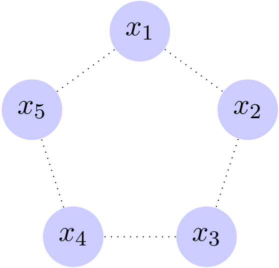
- $I^t \subseteq (L(t))$ [trivial]
- $I^t \supseteq (L(t))$ [difficult]

Graphing Monomials

We graphically represent the monomials by reducing each pair of consecutive vertices to an edge so that we get ***as many edges as we can***.

Consider the monomial $x_1^2 x_2^3 x_3 x_4 x_5^2$:

x_1 x_1
x_2 x_2 x_2
x_3
x_4
x_5



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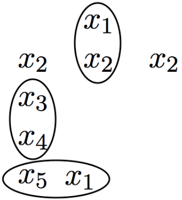

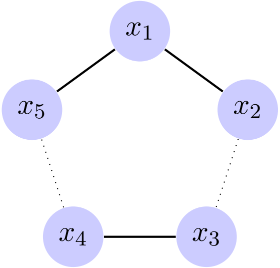

$$x_1^2x_2^3x_3x_4x_5^2 = (x_1x_2)(x_3x_4)(x_1x_5)x_2^2$$

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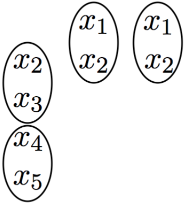
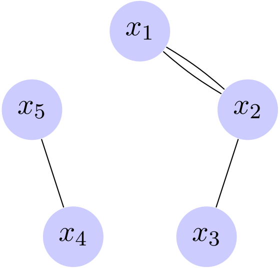


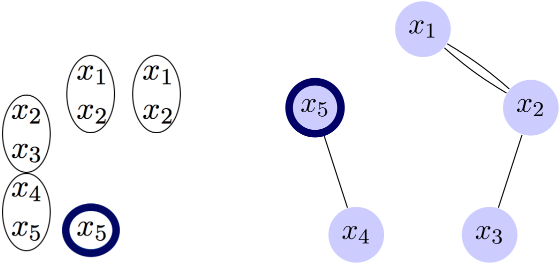


$$x_1^2x_2^3x_3x_4x_5^2 = (x_1x_2)^2(x_2x_3)(x_4x_5)$$

Graphing Monomials

When a monomial cannot reduce perfectly to a product of edges, it creates *ancillary* vertices.

Consider the monomial $x_1^2 x_2^3 x_3 x_4 x_5^2$:

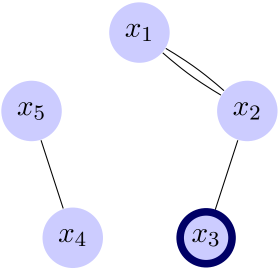


$$x_1^2 x_2^3 x_3 x_4 x_5^2 = (x_1 x_2)^2 (x_2 x_3)(x_4 x_5) x_5$$

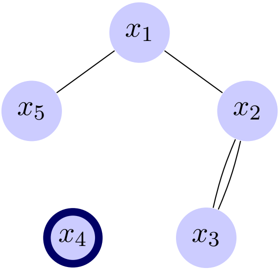
Graphing Monomials

Each monomial does not necessarily correspond to a unique graph.

Consider $x_1^2 x_2^3 x_3^2 x_4 x_5$



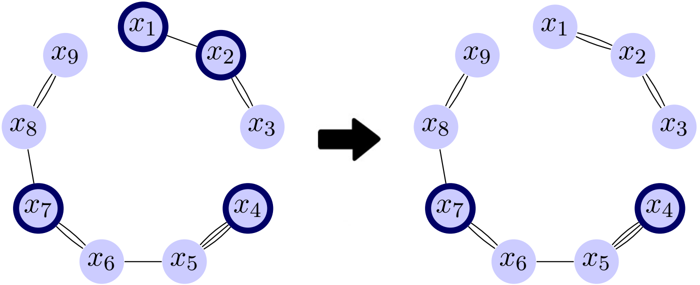
$(x_1x_2)^2(x_2x_3)x_3(x_4x_5)$



$(x_1x_2)(x_2x_3)^2x_4(x_5x_1)$

Graphing Monomials

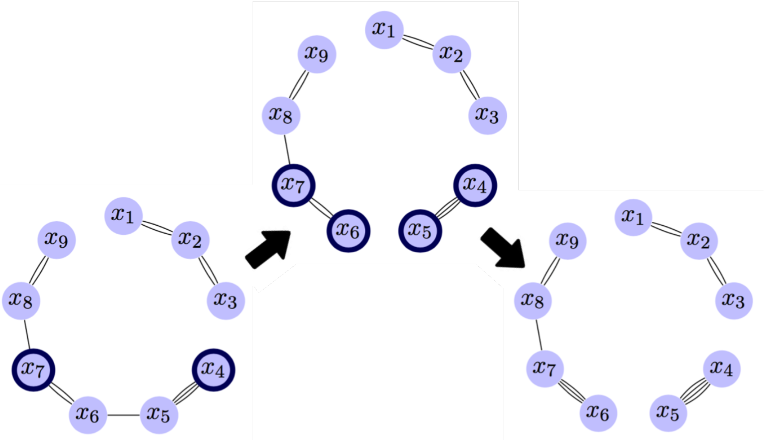
We had to be concerned with an infinite number of possible graphs of varying sizes, so we wanted to understand what graphs can and cannot look like.



Assuming one of each ancillary.

Graphing Monomials

But even after turning those two ancillaries into an edge, it is still not good enough!



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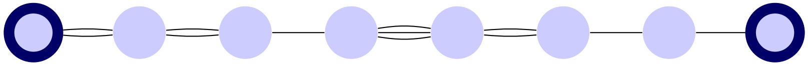
Graphing Monomials



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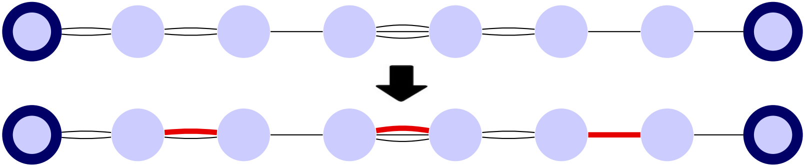

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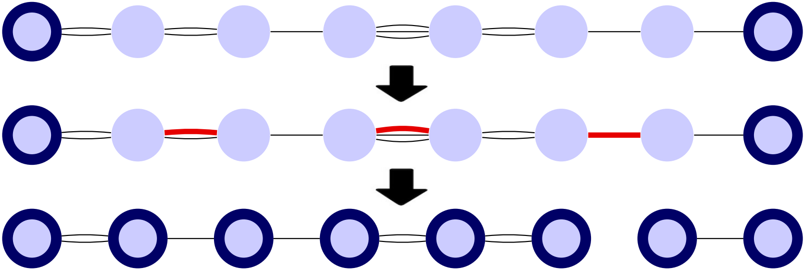

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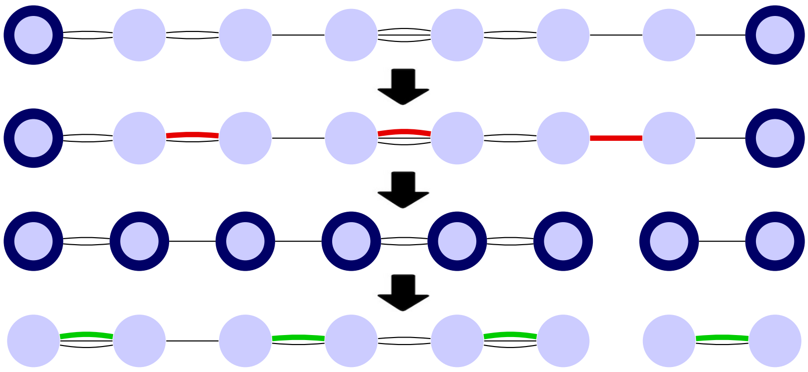

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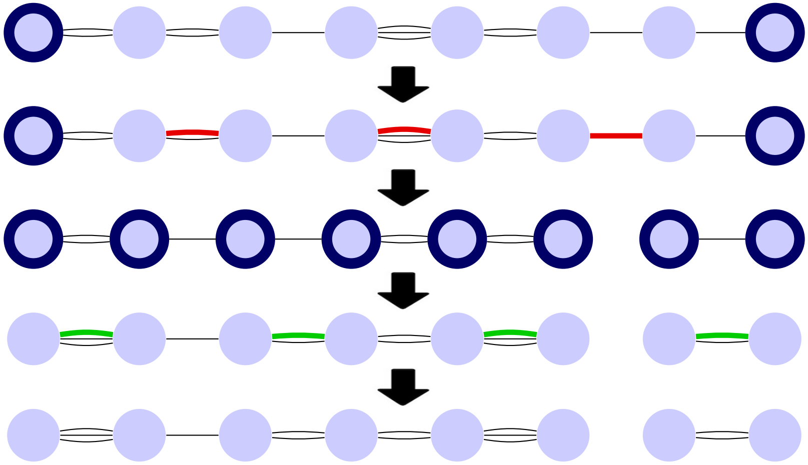
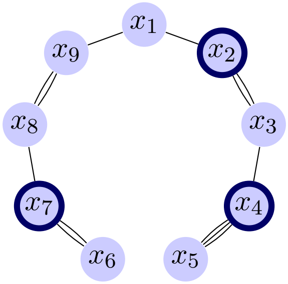


Example:

Let $t = 14$ and let $m \notin I^t$.
Our goal is to demonstrate that $m \notin (L(t))$.

$$m = x_1^2 x_2^{100} x_3^3 x_4^{20} x_5^3 x_6^2 x_7^{54} x_8^3 x_9^3$$



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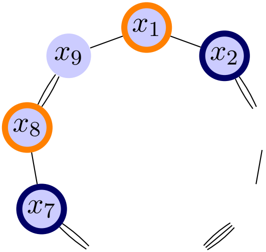

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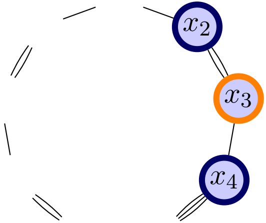

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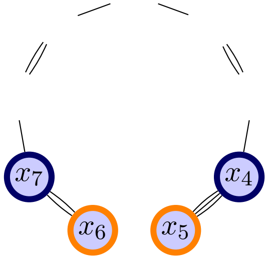

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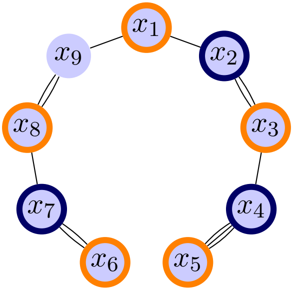

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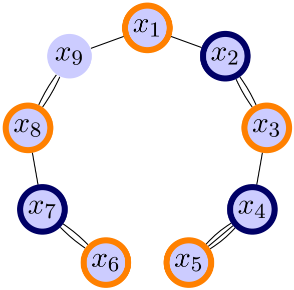

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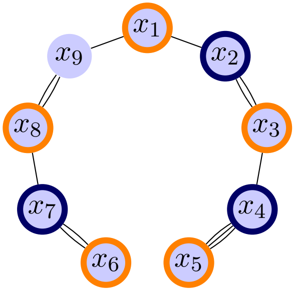

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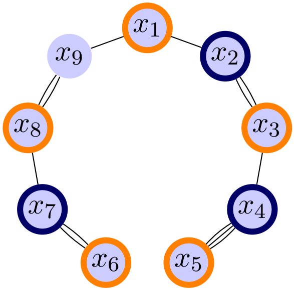

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$m \notin I^t$

$w_V(m) =$ num. edges $< t$

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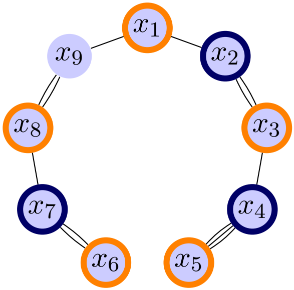

ex: $t = 14$
$m \notin I^t$
$w_V(m) = \text{num. edges} < t$
$m \notin L(t)$

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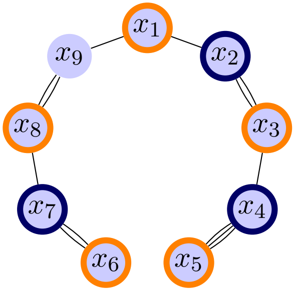

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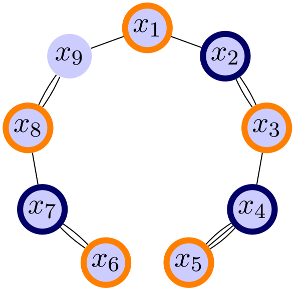

ex: $t = 14$
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$w_V(m) = \text{num. edges} < t$
$m \notin L(t)$
$m \notin (L(t))$
$(L(t)) \subseteq I^t$

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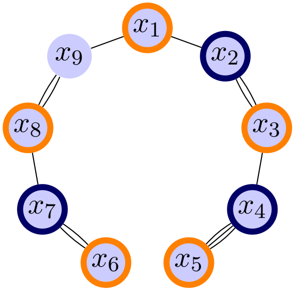

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$m \notin I^t$
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$m \notin (L(t))$
$(L(t)) \subseteq I^t$
$(L(t)) = I^t$

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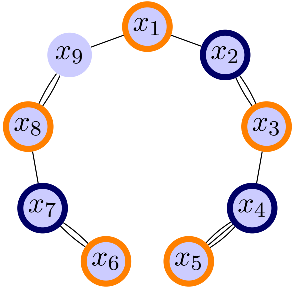

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$w_V(m) =$ num. edges $< t$

$m \notin L(t)$

$m \notin (L(t))$

$(L(t)) \subseteq I^t$

$(L(t)) = I^t$

$I^{(t)} = I^t + (D(t))$

Theorem (Kamp-Vander Woude)

Let G be an odd cycle of size $2n+1$, I be its edge ideal, and V' denote a minimal vertex cover. Then

$$I^{(t)} = I^t + (\{x^{\underline{\alpha}} | \deg(x^{\underline{\alpha}}) < 2t \textit{ and } \forall V' \subseteq V(G),\, w_{V'}(x^{\underline{\alpha}}) \geq t\})$$