Symbolic Powers of Ideals

Thomas Kamp and Jason Vander Woude

Dordt College

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Background of Our Research

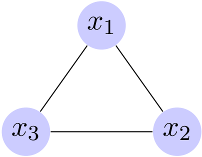
- 1992 *On the Ideal Theory of Graphs*
  - They show that a graph G is bipartite (contains only even cycles) if and only if its corresponding edge ideal satisfies $I^{(n)} = I^n$ for all $n \geq 1$.
- 2004 *Symbolic Powers of Edge Ideals*
  - $I^{(n+r)} \supseteq I^{n+r} + I^{r-1} \cdot \langle x_1 x_2 \cdots x_{2n+1} \rangle$ for all $r \geq 1$

An Introduction to Abstract Algebra and Graph Theory

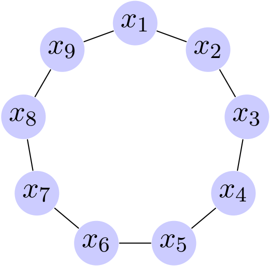
Graphs:

- A collection of vertices and edges connecting those vertices

Cycles:



3-Cycle



9-Cycle

Monomials

Polynomials

$$x^2 + 5x + 6 \qquad 2x^4 + 7x^2 + 3x + 1$$

Multivariable Polynomials

$$x_1^2 x_2 x_3^3 + 5x_1 x_4 + 6x_2 \qquad 2x_1^4 + 7x_1^2 x_4 x_5^3 x_6^7 + 3x_1^5 + 1$$

Monomials

$$x_1^2 x_2 x_3^3 \qquad x_1^2 x_4 x_5^3 x_6^7$$

Ideals

An ideal is a set with special additive and multiplicative properties.

Example:

- The set of all even numbers (all numbers divisible by 2) is an ideal of the integers, and we'll call this ideal E.
  - The sum of any two even numbers is an even number.
  - Any integer times an even number is an even number.
- E^2 is the set of all numbers divisible by $2^2 = 4$.
- E^3 is the set of all numbers divisible by $2^3 = 8$.

Ideals can also be generated from graphs.

Research Goals

- One of the research goals was to prove a pair of conjectures stated in the 2004 paper regarding odd cycles of length $2n+1$.
  - $I^{(t)} = I^t$ for $1 \leq t \leq n$
  - $I^{(n+1)} = I^{n+1} + (x_1 x_2 \cdots x_{2n+1})$
- We have been able to prove both of these conjectures, which has opened up the door for further research for the rest of the summer.

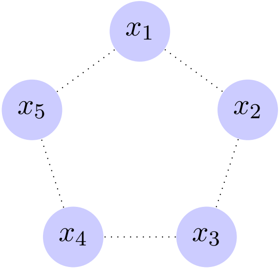
So, what did we actually do?

- Explore background information
- Collect data and explore patterns using Macaulay2
- Formulate and test conjectures until one seems adequate
  - $I^{(t)} = I^t + I_{S\langle t\rangle}$
- Prove the conjecture
  - When is something considered to be true in mathematics?
  - Often requires splitting the conjecture into sub-conjectures and proving them individually or splitting them further
  - Also requires considerable amounts of backtracking
- Write a formal proof in a way that is easy to comprehend
  - Shockingly difficult

Graphing Monomials

We graphically represent the monomials by reducing each pair of consecutive vertices to an edge so that we get ***as many edges as we can***. Consider the monomial $x_1^2 x_2^3 x_3 x_4 x_5$.

x_1	x_1	
x_2	x_2	x_2
x_3		
x_4		
x_5		



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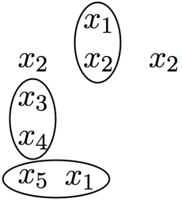

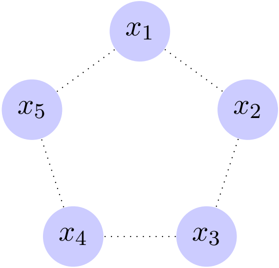

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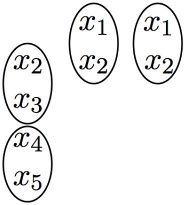
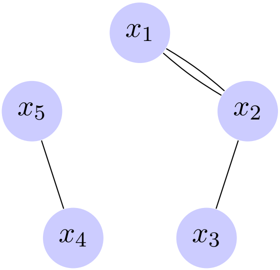
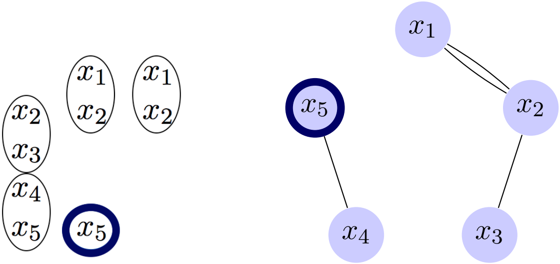




$$(x_1x_2)^2(x_2x_3)(x_4x_5)$$

Graphing Monomials

However, each monomial does not necessarily have to perfectly reduce, which creates *ancillary* vertices. Consider the monomial $x_1^2 x_2^3 x_3 x_4 x_5^2$.

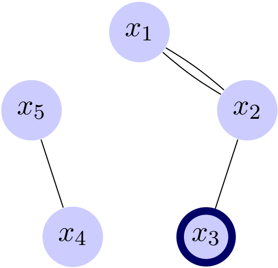


$$(x_1 x_2)^2 (x_2 x_3)(x_4 x_5) x_5$$

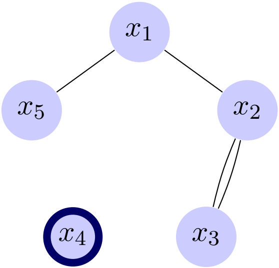
Graphing Monomials

Another problem we encountered was that each monomial does not necessarily correspond to a unique graph.

$$x_1^2x_2^3x_3^2x_4x_5$$



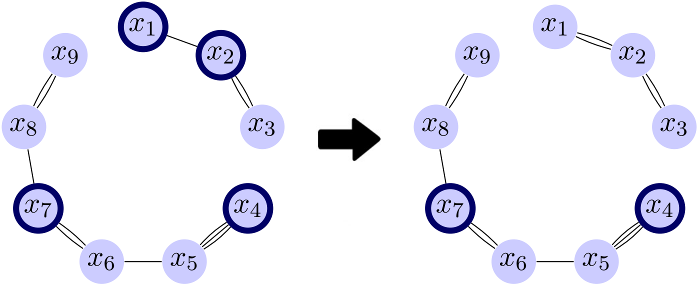
$(x_1x_2)^2(x_2x_3)x_3(x_4x_5)$



$(x_1x_2)(x_2x_3)^2x_4(x_5x_1)$

Graphing Monomials

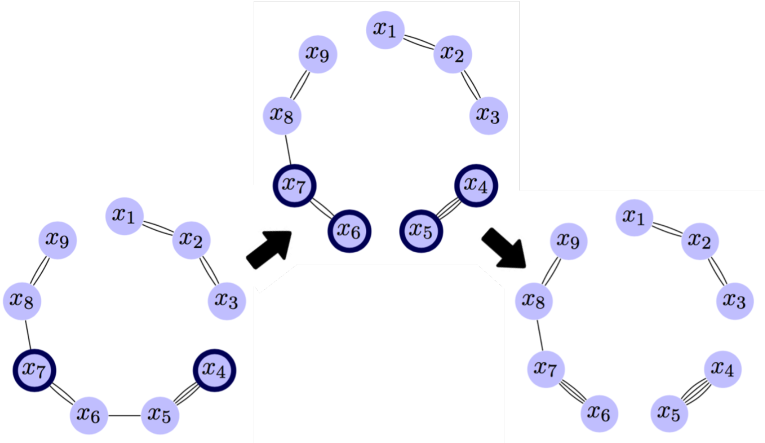
We had to be concerned with an infinite number of possible graphs of varying sizes, so we wanted to understand what graphs can and cannot look like.



Assuming one of each ancillary.

Graphing Monomials

But even after turning those two ancillaries into an edge, it is still not good enough!



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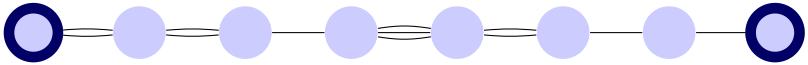
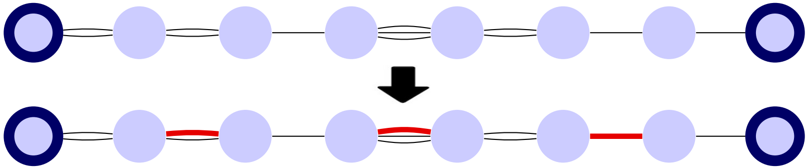
Graphing Monomials



Graphing Monomials



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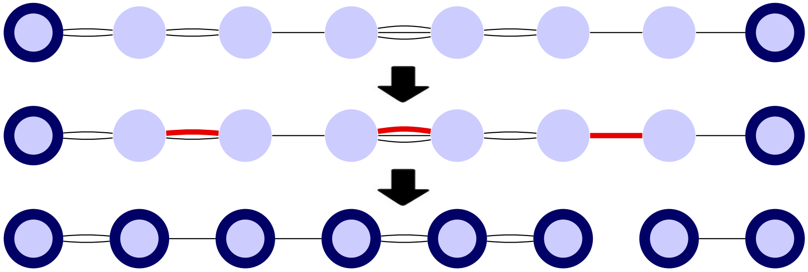

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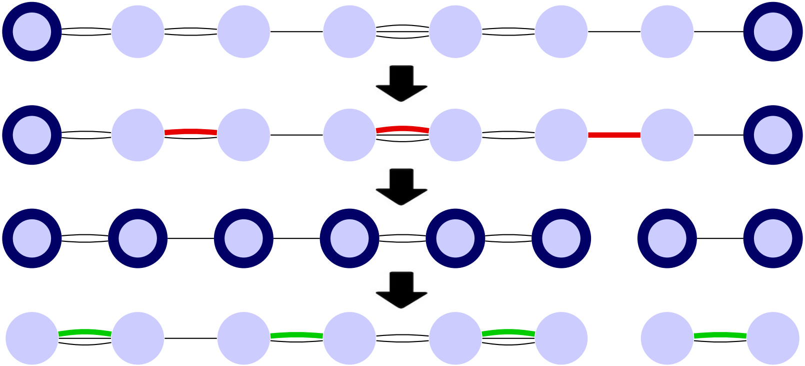
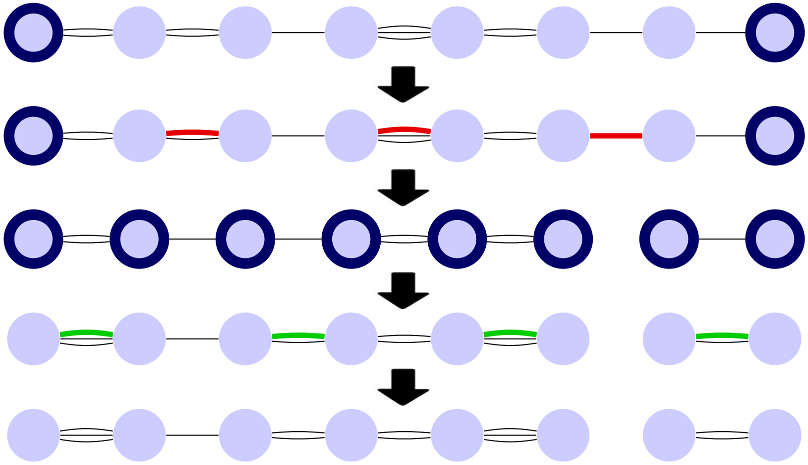


Graphing Monomials



Theorem (Kamp-Vander Woude)

Let G be an odd cycle of size $2n+1$, I be its edge ideal, and V' denote a minimal vertex cover. Then

$$I^{(t)} = I^t + (\{x^{\underline{\alpha}} | \deg(x^{\underline{\alpha}}) < 2t \text{ and } \forall V' \subseteq V(G),\, w_{V'}(x^{\underline{\alpha}}) \geq t\})$$