**A First Course in Differential Equations, 3rd ed.**

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# SOLUTIONS TO EVEN-NUMBERED EXERCISES

This supplement contains solutions, partial solutions, or hints to most of the *even-numbered* exercises in the text. Many of the plots required in the Exercises are not displayed, but rather left to the reader. Solutions to the odd-numbered exercises are posted on the author's web site:
http://www.math.unl.edu/ jlogan1/diff-eqs3.htm

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## Chapter 1 Exercises

■ **Sec. 1.1.2**, **page 9**

2. $x = \tan t$, so $x' = \sec^2 t = 1 + \tan^2 t = 1 + x^2$. Clearly $x(0) = 0$.

6. Let $x = at^2 + bt + c$, so $x' = 2at + b$. Substitute into the DE to get $2at + b + 2(at^2 + bt + c)t^2 + 4t + 7$. Equating like coefficients gives $2a = 1$, $a + b = 2$, $b + 2c = 7$, giving $a = 1/2$, $b = 3/2$, $c = 11/4$.

8. Substitute $x = t^m$ to get $t^2 m(m-1)t^{m-2} - 6t^m = 0$, or $m^2 - m - 6 = 0$. So $m = 3, -2$.

10. Note $x' = C$. So $tx' - x + f(x') = tC - (Ct + f'(C)) + f'(C) = 0$.

12. $I(x) = I_0 e^{-1.4x}$. Setting $I(x) = 0.01 I_0$ and solving for $x$ gives $x = 3.29$ m.

14. $C(t) = 6.68e^{-0.000121t}$. Setting $C(t) = 0.09$ and solving for $t$ gives $t =$ 35596 yrs.

16. In 2050 the population will be approximately 11.5 billion.

■ **Section 1.1.3**, **page 15**

2. Set $x^2 + t^2 = C$. So the isolines are circles of radius $\sqrt{C}$.

4. The nullcline is $t = x^2$, a parabola opening to the right. The slope field is positive when $t > x^2$ (region between the two branches of the parabola) and negative when $t < x^2$ (outside the branches).

## ■ **Section 1.2**, **page 19**

2. $x = \int (t^{1/2} + t^{-1/2})dt = (2/3)t^{3/2} + 2t^{1/2} + C$; $x(1) = 4$ implies $C = 4/3$.

4. **(a)** Use integration by parts: $x(t) = C - (2t+1)/(4\exp(2t))$. **(b)** Use the substitution $y = \ln t$ to get $x(t) = \ln(\ln(t)) + C$. **(c)** Use the substitution $y = \sqrt{t}$.

6. $x(t) = \int_1^t e^{-s}/\sqrt{s}\, ds$.

8. Integrate the equation $x'' = -g$ twice, and evaluate the two constants using the initial conditions. We get $x = -\frac{1}{2}gt^2 + vt + h$.

10. **(a)** Note that the derivative of the erf function is $(2/\sqrt{\pi})e^{-t^2}$. Thus $f'(t) = \text{erf}'(\sin t)\cos t$. **(b)** By the product rule $f'(t) = te^{-\sqrt{t}} + \int_1^t e^{-\sqrt{s}} ds$.

12. Let the initial velocity have $x$ and $y$ components $V\cos\theta$ and $V\sin\theta$. Integrating the $x$ and $y$ equations twice and applying the initial conditions gives $x(t) = (V\cos\theta)t + L$ and $y(t) = -(1/2)gt^2 + (V\sin\theta)t + H$. Maximize the trajectory at the wall, $y(L) = -(1/2)gL^2 + (V\sin\theta)L + H$ with respect to $\theta$. You get $\theta = \pi/2$. So, construct the wall higher than $y(L)$.

14. Pull the $e^{-at}$ factor out of the integral and differentiate using the fundamental theorem of calculus. We obtain a differential equation $x' + (1 + a)x = ab$, with $x(0) = b$. Thus $x(t) = Ce^{-(1+a)t} - ab/(1 + a)$. Use the initial condition to find $C$.

## ■ **Section 1.3.1**, **page 26**

2. $y = Ce^{-rt} + a$.

4. **(a)** $x(t) = C(t + 1)^2$. (b) $\theta(t) = \arcsin\left(\frac{2}{3}(t + 1)^{3/2}\right) + C$. **(c)** Integrate to get $u^2 + u = \frac{1}{2}t^2 + t + C$. Us the quadratic formula to obtain explicit solutions. **(d)** $R(t) = \tan(C + (t + 1)^2/2)$. **(d)** Write the equation as $ydy/(1 + y^2) = -dt$ and integrate to get $\arctan(y^2 + 1) = -2t + C$. **(f)** $x(t) = -1/(C - \ln(t + 1))$.

6. Use the hint to obtain the differential equation $y' = 4 - y^2$. Separate variables and integrate, using partial fractions, to get

$$\ln \frac{|2 - y|}{|2 + y|} = 4t + C, \quad y(0) = -1.$$

   So, $C = \ln 3$. Solve for $y$ to get $y(t) = 2 - 12/(\exp(4t) + 3)$. So $x(t) = 4t - y(t)$, which is valid on $(-\infty, \infty)$.

8. Separate variables to get $e^x dx = t^2 dt$, giving $e^x = (1/3)t^3 + C$. $x(0) = \ln 2$ gives $C = 2$. Then $x(t) = \ln((1/3)t^3 + 2))$. The interval of existence is when $t^3 > -6$, or $t > -3^{1/3}$.

10. **(a)** Separate variables to get $e^{-x}dx = e^t dt$. **(b)** Use partial fractions on the $T$ integral. **(c)** Use the fact $\tan y = \sin y / \cos y$.

12. $y(t) = -1/(\ln(t^2+1) - 1/y_0)$. If $y_0 < 0$ the solution exists for all $t$. If $y_0 > 0$ the solution exists for $\ln(t^2+1) < 1/y_0$.

14. **(a)** The integral curves are ellipses. **(b)** Use implicit differentiation to get $2xx' + 4t = 0$, or $x' = -2t/x$. **(c)** When $t = 1$, $x = 4$ we get $C = 18$.

16. See the formula in the text.

18. By the definitions, $\frac{x'}{x} = r - c_i x = r - \frac{D}{H}x$, which reduces to $x' = rx(1 - x/K)$, where $K = rH/D$.

20. Integrate both sides with respect to $t$ to get $\ln x = a \ln y$, where the constant of integration is set to zero. Then $x = y^a$ by properties of logarithms.

22. Let $y = at + bx + c$, then $y' = a + bx' = a + bF(y)$. (a) Let $y = t + x$; then the equation becomes $y' = 1 + y^2$, giving $y = \tan t$. Then $x = \tan t - t$.

24. Write $(tx')' = -2t$ and integrate to get $tx' = -t^2 + c_1$. Divide by $t$ and integrate again to get $x = -t^2/2 + c_1 \ln t + c_2$.

26. The equation is homogeneous. The solution is $y(t) = ct^2/(1 - Ct)$.

28. Write $-1/y^2 dy = e^{-t^2} dt$ and integrate to get $1/y = \int_0^t e^{-s^2} ds + C = (\sqrt{\pi}/2)\text{erf}(t) + C$. The initial condition gives $C = 2$. Then solve for $y$.

## ■ Section 1.3.2, page 32

2. $T' = -h(T - T_e)$, where $T_e$ is the unknown refrigerator temperature, and $T(0) = 46$. So $T(t) = (46 - T_e)e^{-ht} + T_e$. Now $T(10) = (46 - T_e)e^{-10h} + T_e = 39$, and $T(20) = (46 - T_e)e^{-20h} + T_e = 33$. Now use software to solve the last two equations for $h$ and $T_e$.

4. $T(t) = 58e^{-ht} + 10$. Then $T(9) = 58e^{-9h} + 10 = 57$, giving $h = 0.023$. Then $T(17) \approx 49$ degrees.

## ■ Section 1.3.3, page 34

2. $100C' = 0.5(0.0002 - C)$, $C(0) = 0$. Separate variables and get $C(t) = (1/5000)(1 - \exp(-t/200))$. The equilibrium occurs when $C = 0.0002$.

4. $1000C' = -0.5C$, $C(0) = 5/1000$. Then $C(t) = (1/200)\exp(-t/2000)$.

6. The constant solutions occur when $C' = 0$, or when $qC_{in} - qC - kVC^2 = 0$. Solve the quadratic for $C$ to find physically meaningful constant solutions.

8. The equation is $VC' = -kVC$, $C(0) = C_0$. So $C(t) = C_0 \exp(-kt)$. The residence time $T$ satisfies $T = -(1/k)\ln(0.9)$.

■ **Section 1.4.1**, **page 41**

2. **(a)** $x(t) = C/t^2 + t^2/4$. **(c)** $x(t) = C\exp(-t^2) + t\exp(-t^2)$. **(f)** $x(t) = C(t^2+1)^{-3/2} + 2$.

4. The integrating factor is $\exp(at)$; write the integral with variable upper limit.

6. Write the integrating factor as $\exp(\int_1^t e^{-s}/s\,ds)$.

8. Write the equation as $y' = 1 + x' = 1 + y^2$, so $y(t) = \tan(t + C)$.

10. The integrating factor is $e^{-pt}$. So, $(xe^{-pt})' = e^{-pt}q(t)$. Integrate from $t_0$ to $t$ to get $x(t) = x_0 e^{pt} + e^{pt}\int_{t_0}^t e^{-ps}q(s)ds$.

12. $(x_1/x_2)' = (x_2x_1' - x_2x_1')/x_2^2$. The numerator is zero because $x_1$ and $x_2$ satisfy the equation, and so $x_1/x_2 = C$, a constant.

14. In changing variables $y = x^{1-n}$, note that $y' = (1-n)x^{-n}x'$, where the chain rule is used.

15. **(e)** Let $y = x^{-2}$, so $y' = -2x^{-3}x'$. Writing the equation in terms of $y$ gives $y' = -2ay - 2b$, which is linear. Clearly $y = Ce^{-2at} - b/a$. Thus $x = 1/\sqrt{y} = 1/\sqrt{C\exp(-2at) - b/a}$.

16. If $H_t = f$ and $H_x = g$, then $H_{tx} = f_x = H_{xt} = g_t$. (ii) $f_x = 1/t$ and $g_t = 1/t$, so the equation is exact. Now find $H$. Note $H_t = t^3 + x/t$ and $H_x = x^2 + \ln t$. Integrating the first gives $H = (1/4)t^4 + x\ln t + \phi(x)$. Now $H_x = x^2 + \ln t = \phi'(x) + \ln t$. So $\phi'(x) = x^2$, or $\phi(x) = (1/3)x^3$. Then the integral curves are $H(t,x) = (1/4)t^4 + x\ln t + (1/3)x^3 = C$.

■ **Section 1.4.2**, **page 46**

2. Constant solutions, or equilibria, are found by setting $S' = 0$, or $(a + rA/M)S = rA$. So $S = rA/(a + rA/M)$.

4. $T' = -3(T - (9 + 10\cos(2\pi t)))$. The solution is

$$T(t) = 3e^{-3t} + 9 + \frac{1}{4\pi^2 + 9}(90\cos(2\pi t) - 90e^{-3t} + 60\pi\sin(2\pi t)).$$

6. For $t < T$ the equation is $S' = -(a + rA/M)S + rA$, and for $t > T$ the equation is $S' = -aS$. The solution to the first is $S_1(t) = C_1 e^{-(a+rA/M)t} + rA/(a + rA/M)$, $t < T$. The solution to the second is $S_2(t) = C_2e^{-at}$, $t > T$. Use the conditions $S_1(0) = S_0$ and $S_1(T) = S_2(T)$ to obtain $C_1$ and $C_2$.

8. Note that $S' = I(1 - S/P) - ES/P$. **(a)** Set $S' = 0$ and solve for $S$ to get the longtime number of species. **(b)** The equation is linear, so use an integrating factor.

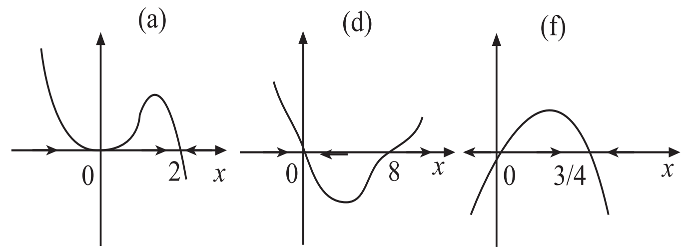


Figure 1: Phase line diagrams for Sec. 1.5.1, Exercises 2adf.

10. The equation becomes $VC' = -qC - kVC^2$, which is a Bernoulli equation. To solve, make the transformation $y = C^{-1}$, giving $y' = -C^{-2}C'$. Write the linear equation for $y$ as $Vy' = 2qy + 2kV$; the solution is $y = c_1e^{2qt/V} - kV/q$. Then $C = 1/y$.

12. Both equations are Bernoulli equations and can be solved by using the transformation $y = X^{-1}$, as in Exercise 10.

■ **Section 1.4.3**, **page 53**

2. The equation is $20Q' + 100Q = 200e^{-5t}$, $Q(0) = 0$. Solving, $Q(t) = 10t\exp(-5t)$. To find the maximum charge set $Q'(t) = 0$ and solve for $t$ to get $t = 1/5$.

4. The circuit equation is $LI'' + RI' + (1/C)I = E'(t)$. By Kirchhoff's law $I'(0) = (1/L)(E(0) - RI_0 - Q_0/C)$.

6. The equation is $LI'' + RI + I/C = -A\omega \sin \omega t$. The initial conditions are $I(0) = I_0$, and $I'(0)$ given by Exercise 4.

8. $I'' + \frac{1}{2}\left(\frac{1}{3}I^3 - I\right) + I = 0$.

■ **Section 1.5.1**, **page 63**

2. **(a)** $x' = x^2(2 - x)$. Equilibria are $x = 0, 2$. For stability check $f_x(x) = 2x(2 - x) - x^2$. Now $f_x(2) = -4 < 0$, so $x = 2$ is stable. $f_(0) = 0$ so at $x = 0$ there is no information. The phase line plot in Figure 1 shows it is unstable. **(d)** $x' = x(x - 8)^3$. The equilibria are $x = 0, 8$. For stability, check $f_x(x) = 3x(x - 8)^2 + (x - 8)^3$. Now, $f_x(0) = -8^3 < 0$, so $x = 0$ is stable. $f_x(8) = 0$, so at $x = 8$ there is no information. But the phase line plot in Figure 1 shows it is unstable. **(f)** $x' = 2x(1 - x) - \frac{1}{2}x$. The equilibria are $x = 0$, $x = 3/4$, which are unstable and stable, respectively.

4. The equilibria are $P = 0$, $a$, $K$ which are stable, unstable, and stable, respectively. If $P < a$, i.e., $P$ is small, then the population becomes extinct for lack of mating partners; if $a < P < K$ or $P > K$, the population approaches the carrying capacity $K$. See Figure 2.

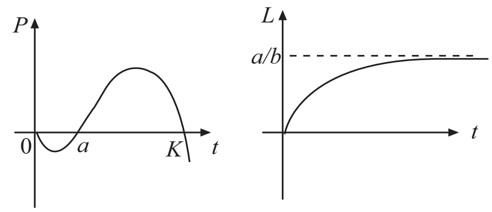


Figure 2: Phase line diagrams for Sec. 1.5.1, Exercises 4 (left) and 8 (right).

6. The function $f(T)$ at the equilibrium has negative slope at the equilibrium $T = S$ which is stable. The approximation leads to the differential equation $T' = -kT^4$, which has solution $T(t) = [3kt + 2000^{-3}]^{-1/3}$.

8. If $L$ is the length and $V = L^3$ is the volume, then $V' = 3L^2L'$ and the growth equation is $V' = \alpha(6L^3) - \beta x^3$, or $L' = a - bL$ for constants $a$ and $b$. If $L(0) = 0$ the solution is $L(t) = \frac{a}{b}(1 - \exp(-bt)$. As $t \to \infty$, $L \to a/b$, consistent with the fact that organisms reach a maximum length. See Figure 2.

10. $I' = aSI = aI(N - I) = aNI(1 - I/N)$, which is the same as the logistic equation. As $t$ gets large the solution approaches $N$, so everyone eventually gets the disease.

12. $P' = \alpha\left(\frac{a}{P} - bP\right) = \alpha\left(\frac{a-bP^2}{P}\right)$. The equilibrium price is $P = \sqrt{a/b}$ and it is stable. Note the graphs of $a/P$ and $bP$ must intersect at the equilibrium. To solve the differential equation separate variables and write $PdP/(a - bP^2) = \alpha dt$; use substitution to calculate the integral.

14. For part (a), we use the chain rule to get

$$\frac{d}{dt}V(x(t)) = V'(x(t))x'(t) = -f(x(t))f(x(t)) < 0.$$

For (b), if $V$ has a local minimum at $x^*$, then in a small neighborhood of $x^*$, $V'(x) \leq 0$ for $x < x^*$ and $V'(x) \geq 0$ for $x > x^*$. Thus $f(x) > 0$ for $x < x^*$ and $f(x) < 0$ for $x > x^*$. So, $f'(x^*) < 0$, and $x^*$ is stable.

■ **Section 1.5.2**, **page 70**

1. (d) The equilibria are given by $x = 1$ and $x = \pm\sqrt{h} > 0$. See Figure 3.

2. $x = 0$ is the only equilibrium and $f_x(0) = 1 > 0$, so $x = 0$ is unstable.

4. The equilibria are $x = 0$ and $x = h - 1$. Note $h > 0$ and $x \geq 0$. Using the derivative criterion, $x = 0$ is unstable for $h < 1$ and stable for $h > 0$; $x = h - 1$ is unstable.

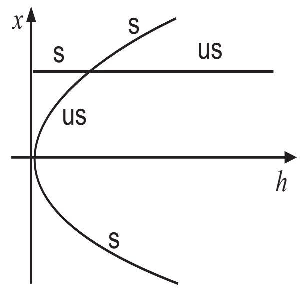


Figure 3: Bifurcation diagram for Exercise 1d, Sec. 1.5.2, showing stable (s) and unstable (us) branches.

6. $x' = x^3 - x + h$. Setting $x' = 0$ gives the parameter $h = x - x^3 = x(1-x)(1+x)$ in terms of the equilibria $x$. This is an easy to plot cubic. One can rotate the graph through 90 degrees to get $x$ vs. $h$, but it is not necessary. To check stability, note $f_x(x) = 3x^2 - 1 > 0$ when $x > 1/\sqrt{3}$ or $x < -1/\sqrt{3}$; These equilibria are unstable. $x$ is stable if $f_x(x) = 3x^2 - 1 < 0$, or $-1/\sqrt{3} < x < 1/\sqrt{3}$.

8. Note $a > 0$ is necessary to have an equilibrium, which are clearly given by $x = \pm 1/\sqrt{a}$. These upper and lower branches of the equilibrium curves are unstable and stable, respectively, because $f_x(x) = 2ax$.

10. **(c)** See Figure 4. Equilibria are found graphically at intersection points of $p-a$ and the sigmoid curve $\rho p^n/(1+p^n)$. If $p-a > \rho p^n/(1+p^n)$ the phase arrow it positive (to the right); if $p - a < \rho p^n/(1+p^n)$ the phase arrow it positive (to the left). This determines the stability of the equilibrium or intersection point.

■ **Section 1.5.3**, **page 76**

2. $f(t,x)$ and $f_x(t,x)$ are are continuous everywhere *except* on the vertical lines $t = \pm 1$. Therefore the problem has a unique solution in an interval contained in $(-1,1)$.

4. **(a)** The problem has singularities at $t = 0, 5$. The initial condition is given at $t = 2$. The largest interval where a solution exists is $(0,5)$. **(b)** The largest interval where a solution exists in $(3,7)$.

6. **(a)** The implicit solution to $x' = -4t/x,\ x(0) = x_0$, is $x^2 + 4t^2 = C$, which is an ellipse in the $tx$ plane. Now, $x(0) = x_0$ gives $x^2 + 4t^2 = x_0^2$, or $x = \pm\sqrt{x_0^2 - 4t^2}$. If $x_0 > 0$ then the solution is the upper branch of the ellipse (with the + sign), and if $x_0 < 0$ the solution is the bottom portion of the ellipse (with the - sign). **(c)** The solution is $x = 1/(1/x_0 - t)$. If $x_0 > 0$ the solution is valid on $(-\infty, 1/x_0)$; if $x_0 < 0$ the solution is valid on $(1/x_0, \infty)$.

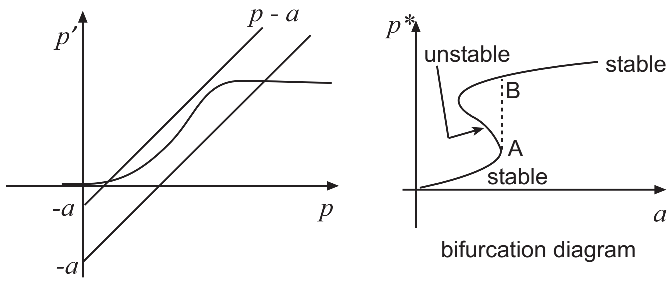


Figure 4: Exercise 1.5.2 Exercise 10. *Left*: Plots of $p - a$ for two values of $a$ and the sigmoid curve $\rho p^n/(1 + p^n)$. Equilibria are at intersection points. As $a$ increases from a small to large value the straight line $p - a$ moves downward; at first a small equilibrium, then a bifurcation occurs and two equilibria appear at the upper tangency point; then there are three, then two at the lower tangency point, and then one at the upper intersection. *Right*: Bifurcation diagram showing how $p^*$ changes as $a$ increases. The upper and lower branches are stable and the middle branch is unstable. As the inflow rate $a$ increases and the equilibrium reaches the point A, it jumps quickly to the upper branch at B (dashed) leading to eutrophication or high phosphorus content.

8. Both parts of the function clearly satisfy the differential equation and the initial condition. But the function $f(t, x)$, as well as $f_x(x, t)$, are not continuous along $t = 0$, where the initial condition is given. Because the problem does not satisfy the hypotheses of the theorem, nothing can be said one way or the other. There is no contradiction.

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## Chapter 2 Exercises

■ **Sec. 2.1, page 84**

2. Multiply the equation by $Q'$ to get $LQ'Q'' + RQ'Q' + \frac{1}{C}QQ' = 0$, which by the chain rule is the same as

$$\frac{d}{dt}\left(\frac{1}{2}L(Q')^2 + \frac{1}{2C}Q^2\right) = -RQ'Q'.$$

The energy in the inductor is $\frac{1}{2}L(Q')^2$ or $\frac{1}{2}LI^2$, and the energy on the capacitor is $\frac{1}{2C}Q^2$; the power lost is $-RI^2$. If there is no resistor then energy is conserved.

■ **Sec. 2.2.2, page 90**

2. (a) $x(t) = e^{2t} - 2te^{2t}$.

   (c) $x(t) = e^{-t}(1 + t)$.

   (d) $x(t) = \frac{3}{2}e^{-t} - \frac{1}{2}e^{-3t}$.

4. Only (c) and (d), because a physical application to oscillators and circuits require nonnegative coefficients.

6. The eigenvalues are 4 and $-6$ so the characteristic polynomial must be $(\lambda-4)(\lambda+6)=0$, or $\lambda^2+2\lambda-24=0$. The equation is $x''+2x'-24x=0$.

8. The general solution is $x(t)=(A+Bt)e^{-at}$. Assume that $x(t_1)=x(t_2)=0$, $t_1\neq t_2$. Then $x(t_1)=A+Bt_1=x(t_2)=A+Bt_2=0$. Therefore, $t_1=t_2$, a contradiction.

■ **Sec. 2.2.3, page 94**

2. (a) $x(t)=e^{-t/2}\left(\cos\frac{\sqrt{15}\,t}{2}+\frac{1}{15}\sin\frac{\sqrt{15}\,t}{2}\right)$.

   (b) $x(t)=\exp(2t)\left(\cos\sqrt{2}\,t-\sqrt{2}\sin\sqrt{2}\,t\right)$.

   (c) $x(t)=\cos 3t$.

4. The roots of the characteristic equation must be $\pm 5$ so $\lambda^2-25=0$, giving $x''-25x=0$. Initial conditions give $c_1=2,\ c_2=0$.

6. It involves the 5 'most famous' numbers 0, 1, $e$, $\pi$, $i$.

■ **Sec. 2.2.4, page 99**

2. $x(t)=e^{-t/16}\left(2\cos((\sqrt{255}/16)t)+\frac{2\sqrt{255}}{255}\sin((\sqrt{255}/16)t)\right)$. The amplitude is $A=2.004$ , the frequency is $\sqrt{255}/16$, and the phase is $\phi=0.0625$.

4. The eigenvalues are $\lambda=\frac{1}{2L}(-1\pm\sqrt{1-4L})$. If $L>1/4$ the the roots are complex giving a decaying oscillation. If $L<1/4$ the roots are real and negative giving decay.

6. The circuit equation is $5I''+\frac{1}{2}I=0$. The solution is $I(t)=\cos\frac{t}{\sqrt{10}}+\sqrt{10}\cos\frac{t}{\sqrt{10}}$. The circuit oscillates. We have $A=\sqrt{11}$, $\omega=1/\sqrt{10}$, and $\phi=1.26$, giving
$$x(t)=\sqrt{11}\cos\left(\frac{t}{\sqrt{10}}-1.26\right).$$

■ **Sec. 2.3.1, page 110**

1. (a) $At^2+Bt+C$ (e) $A\sin 7t+B\cos 7t$. (e) $At+B+Ce^{-t}$.

2. (a) $x(t)=c_1\cos(\sqrt{7}\,t)+c_2\sin(\sqrt{7}\,t)+\frac{t}{16}\exp(3t)-\frac{1}{128}\exp(3t)$.

   (b) $x(t)=c_1+c_2e^t+\frac{1}{2}e^{2t}-6-6t$.

   (f) $x(t)=c_1e^{-t}+4te^{-t}$.

   (g) $x(t)=c_1e^{2t}+c_2e^{-2t}-\frac{1}{8}\cos 2t$.

4. $x(t)=\frac{11}{117}e^{8t}-\frac{1}{26}e^{-5t}-\frac{1}{18}e^{-t}$.

6. The differential equation is $x'' + 2x = \cos\sqrt{2}\,t$. The solution with initial conditions is
$$x(t) = \frac{1}{4}(t+2)\sqrt{2}\sin\sqrt{2}\,t.$$

8. $y(t) = \frac{5}{2} - e^{-t} - \frac{1}{2}e^{-2t}$.

10. The circuit equation is $0.4Q'' + 100Q = 5e^{-2t}$, $Q(0) = Q'(0) = 0$. The charge is
$$Q(t) = \frac{25}{508}e^{-2t} - \frac{25}{508}\cos(5\sqrt{10}\,t) + \frac{\sqrt{10}}{508}\sin(5\sqrt{10}\,t).$$
The first term is the transient that decays away leaving the final two terms, which represent an oscillatory steady state.

■ **Sec. 2.3.2, page 114**

2. Using the initial conditions we get $c_1 = 0$ and $c_2 = -B/\omega_0$.

4. The circuit equation is $LQ'' + \frac{1}{C}Q = V_0 \sin\beta t$. Resonance occurs when the natural frequency equals $\beta$, or
$$\sqrt{\frac{1}{LC}} = \beta \quad \text{or} \quad L = \frac{1}{\beta^2 C}.$$

6. The solution is
$$x(t) = 50\sin 2t - e^{t/200}\left(50.0002\sin 1.999t\right).$$

■ **Sec. 2.4.1, page 120**

1. (b) The indicial equation is $m^2 - m - 4 = 0$ having real roots $\frac{1}{2}(1 + \sqrt{17})$. The general solution is $x(t) = c_1 t^{1/2+\sqrt{17}/2} + c_2 t^{1/2-\sqrt{17}/2}$.

   (e) The characteristic equation is $m^2 - 8m + 16 = 0$ having roots $4 \pm 2i$. The general solution is $x(t) = t^4(c_1\cos(2\ln t) + c_2\sin(2\ln t)))$.

2. This is not a Cauchy-Euler equation. Hence, let $y = x'$ so that $y' + t^2 y = 0$. This equation is separable and has solution $y(t) = \exp(-t^3/3)$, where we have used $y(0) = 1$. Therefore $x(t) = \int_0^t \exp(-r^3/3)dr$.

4. The equation can be written $(a(t)x')' = f(t)$, which when integrated gives
$$x'(t) = \frac{1}{a(t)}\int_\alpha^t f(r)dr + c_1\frac{1}{a(t)}.$$
Integrating again,
$$x(t) = \int_\alpha^t \left(\frac{1}{a(s)}\int_\alpha^s f(r)dr\right)ds + c_1\int_\alpha^t \frac{1}{a(s)}ds + c_2.$$

■ **Sec. 2.4.2**, **page 124**

2. This is a Cauchy-Euler equation. The indicial equation for the homogeneous equation is $m^2 = 0$, so $m = 0, 0$, giving independent solutions $x_1(t) = 1$ and $x_2(t) = \ln t$. The Wronskian is $t^{-1}$. Therefore the particular solution is

$$x_p(t) = -\int \frac{a \ln t}{t^{-1}} dt + \ln t \int \frac{1 \cdot a}{t^{-1}} dt = \frac{a}{4} t^2.$$

The general solution is $x(t) = c_1 + c_2 \ln t + \frac{a}{4} t^2$.

4. (a) We have

$$\begin{aligned} W'(t) &= (x_1 x_2' + x_1' x_2)' = x_1 x_2'' - x_2 x_1'' \\ &= x_1(-p x_2' - q x_2) - x_2(-p x_1' - q x_1) = -pW(t). \end{aligned}$$

(b) Separate variables to get $dW/W = -p(t)dt$ and integrate to get $W(t) = C \exp(-\int p(t)dt)$. If $C = 0$ then $W(t) \equiv 0$ for all $t$, and if $C \neq 0$, then $W(t)$ is never zero.

(c) Assume $x_1(a) = x_1(b) = 0$. Then $x_1'(a)$ and $x_1'(b)$ have opposite signs. Now, $W(t) = x_1(t)x_2'(t) - x_1'(t)x_2(t)$; so that $W(a) = -x_1'(a)x_2(a)$ and $W(b) = -x_1'(b)x_2(b)$. These two expressions must have the same sign, so $x_2(a)$ and $x_2(b)$ must have opposite signs. Therefore $x_2(t)$ must equal zero at some point between $a$ and $b$.

6. $x_p(t) = e^{mt} \int_0^t (t - s)e^{-ms} f(s)ds$.

■ **Sec. 2.4.3**, **page 125**

2. Let $x_2 = tv$ and substitute into the equation to get $tv'' + (2 - t^2)v' = 0$. Letting $y = v'$ we get a separable equation $dy/y = (t - 2/t)dt$. Integrating, we get $\ln y = \frac{1}{2}t^2 - 2 \ln t$. Hence,

$$y(t) = \frac{1}{t^2} e^{t^2/2}, \quad \text{giving} \quad x_2(t) = t \int \frac{1}{t^2} e^{t^2/2} dt.$$

Note: this solution can be written in different ways using special functions defined by integrals.

5. Let $x_2 = tv$. then $x_2' = tv' + v$ and $x_2'' = tv'' + 2v'$. The equation reduces to $v'' - v' = 0$, having solution $v = e^t$. Therefore, a second independent solution is $x_2(t) = te^t$.

7. Hint: to solve the equation for $z$ separate variables and integrate.

8. In Exercise 7 let $y(t) = e^t$; we know $z(t)e^t = x_2(t) = \cos t$. Thus $z(t) = e^{-t} \cos t$. Substitute this into the $z$ equation to get $p(t) = (\sin t - \cos t)/\cos t$. Return to the original equation for $x$ and then compute $q(t) = -1 - p(t)$.

■ **Sec. 2.5**, **page 130**

1. (b) The characteristic equation is $\lambda^4 + \lambda = \lambda(\lambda^3 + 1) = 0$. The second factor has root $\lambda = -1$ and so the characteristic polynomial turns into $\lambda(\lambda + 1)(\lambda^2 - \lambda + 1) = 0$, having roots $\lambda = 0, -1, \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$. The general solution to the homogeneous equation is

$$x_h(t) = c_1 + c_2e^{-t} + e^{t/2}(c_3 \cos \sqrt{3}\,t/2 + c_4 \sin \sqrt{3}\,t/2).$$

   We guess a particular solution as $x_p(t) = At$ which upon substitution gives $A = 1$. Thus, $x(t) = x_h(t) + t$.

2. The characteristic equation is $\lambda^3 + \lambda^2 - 4\lambda - 4 = 0$. Easily, $\lambda = -1$ is a root so the polynomial factors into $(\lambda+1)(\lambda^2 - 4) = 0$, giving eigenvalues $-1, 2, -2$. The general solution is therefore $x(t) = c_1e^{-t} + c_2e^{2t} + c_3e^{-2t}$. Applying the initial conditions gives three equations with three unknowns $c_1$, $c_2$, $c_3$, and we find that $c_1 = 5/3$, $c_2 = 1/12$, $c_3 = -3/4$.

4. The characteristic polynomial is $(\lambda^2+1)(\lambda^2+1) = 0$ giving the differential equation $x'''' + 2x'' + x = 0$. The general solution is $x(t) = A\cos t + B\sin t + t(C\cos t + D\sin t)$.

■ **Sec. 2.6**, **page 135**

2. $u(x) = -x^3/6 + x^4/240 + 100x/3$. The rate heat leaves the right end is $-Ku'(20)$.

4. There are no nontrivial solutions for $\lambda \le 0$. When $\lambda = n^2\pi^2$, $n = 1, 2, 3, \ldots$ we get solutions $u_n(x) = \sin n\pi x$.

6. The characteristic equation is $p^2 + 2p + \lambda = 0$ with roots $p = -1 \pm \sqrt{1-\lambda}$. If $\lambda \le 1$ then the solutions are exponential and cannot satisfy the boundary conditions; so $\lambda > 1$. The roots are then $p = -1 \pm \sqrt{\lambda - 1}\,i$. Therefore $u(x) = e^{-x}(A\cos(\sqrt{\lambda-1}\,x) + B\sin(\sqrt{\lambda-1}\,x)$. Now, $u(0) = 0$ implies $A = 0$. Then $u(x) = Be^{-x}\sin(\sqrt{\lambda-1}\,x)$. Apply the right boundary condition to get $u(1) = Be^{-1}\sin(\sqrt{\lambda-1}) = 0$. Therefore $\sqrt{\lambda-1} = n\pi$, $n = 1, 2, , 3, \ldots$. Thus the eigenvalues are $\lambda_n = 1 + n^2\pi^2$ with $u_n(x) = e^{-x}\sin n\pi x$, $n = 1, 2, 3, \ldots$.

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## Chapter 3 Exercises

■ **Section 3.1**, **page 144**

2. $F(s) = \int_0^\infty e^{-3t}H(t-2)e^{-st}dt = \int_2^\infty e^{-(3+s)t}dt = \frac{1}{3+s}e^{-2(3+s)}$.

6. $\mathcal{L}[f(t)H(t-a)] = \int_a^\infty f(t)e^{-st}dt = \int_0^\infty f(r+a)e^{-(r+a)s}ds = e^{-as}\mathcal{L}f(r+a)$.

8. (a) $\frac{6}{s} + \frac{5}{s+2} + \frac{1}{s-3)^2}$. (c) $\cos(5t)$. (e) $\frac{3}{2}\left(\frac{1}{s} + \frac{2}{s+1}\right)$.

9. (c) $2t^2e^{5t}$. (d) $7H(t-4)$. (g) $\frac{3\sqrt{2}}{2\sqrt{7}}\sin\sqrt{\frac{7}{2}}t$.

10. (b) Use $\sin t{-}\pi/2) = -\cos t$. (c) $\mathcal{L}[H(t{-}\pi/2)\sin(t{-}\pi/2)] = e^{-\pi s/2}\frac{1}{s^2+1}$.

12. By the shift property, $X(s) = \frac{1}{s+1}e^{-2(s+1)}$.

14. $x(t) = e^{t^2}$ is not of exponential order, i.e., it is not bounded by $Me^{rt}$ for any $r$. The improper integral $\int_0^\infty \frac{1}{t}e^{-st}dt$ does not exist at $t = 0$. $X(s) = \frac{1}{s}e^s$ does not go to zero as $s \to \infty$ so its transform does not exist.

16. Letting $st = r^2$, so that $sdt = 2rdr$ gives

$$\int_0^\infty \frac{1}{\sqrt{t}}e^{-st}dt = 2\int_0^\infty e^{-r^2}dr = \sqrt{\pi}.$$

18. Let $\mathcal{L}f = F$ and $\mathcal{L}g = G$. Then

$$\begin{aligned}\mathcal{L}^{-1}[aF + bG] &= \mathcal{L}^{-1}[a\mathcal{L}f + b\mathcal{L}g] \\ &= \mathcal{L}^{-1}\mathcal{L}[af + bg] = af + bg = a\mathcal{L}^{-1}F + b\mathcal{L}^{-1}G.\end{aligned}$$

## ■ Section 3.2, page 156

2. (a) Use $\cos t = -\cos(t-\pi)$. Then $-\mathcal{L}[H(t-\pi)\cos(t-\pi)] = -\frac{s}{s^2+1}e^{-\pi s}$.
   (b) Use the Table to get $3 \cdot 4!/s^5$.
   (e) Use the Table to get $3/((s+6)^2+9)$.
   (f) $\int_0^\infty e^{a+bi-s)t}dt = -1/(a-s+bi)$ provided $s > a$.

4. In multiline form

$$\begin{cases} 0, & t < 1; \\ t-1, & 1 \le t < 3; \\ t-3, & 3 \le t < 4; \\ t-3+e^{-t/2}, & t \ge 4. \end{cases}$$

6. (b) The differential equation transforms to $X(s) = \frac{1}{s+1}\frac{2}{s^2+4}$. Using partial fractions we get $x(t) = \frac{1}{5}(e^{-t} - 2\cos 2t + \sin 2t)$.
   (e) The differential equation transforms to $X(s) = \frac{1}{(s-1)^2+1} + \frac{1}{s+1}\frac{1}{(s-1)^2+1}$. Using partial fractions $x(t) = \frac{1}{5}e^{-t} - \frac{1}{5}e^t(\cos t + 7\sin t)$.
   (i) The differential equation transforms to $X(s) = \frac{s}{s^2-2} + \frac{1}{s}\frac{1}{s^2-2}$. Partial fraction gives $x(t) = \frac{3}{4}(\exp(\sqrt{2}\,t) + \exp(-\sqrt{2}\,t) - \frac{1}{2}$.

8. $\mathcal{L}[t^2H(t-3)] = e^{-3s}\mathcal{L}[(t+3)^2] = e^{-3s}(2//s^3 + 6/s^2 + 9/s)$.

10. $f(t) = t - (t-4)H(t-4)$.

12. Taking Laplace transforms of the equation gives $X(s) = \frac{1}{s-1} - \frac{2}{s(s-1)}e^{-s}$. Inverting, $x(t) = e^t - H(t-1)(e^{t-1} - 1)$.

14. Taking Laplace transforms of the equation gives $X(s) = \frac{1}{s+1} + \frac{1}{s(s+1)}e^{-s} - \frac{1}{s(s+1)}e^{-2s}$. Inverting gives $x(t) = e^{-t}e^{1-t}H(t-1) - e^{2-t}H(t-2)$.

16. Write

$$\begin{aligned}F(s) &= \int_0^\infty f(t)e^{-st}dt = \sum_0^\infty \int_{np}^{(n+1)p} f(t)e^{-st}dt \\ &= \sum_0^\infty \int_0^p f(r+np)e^{-s(r+np)}dr = \sum_0^\infty \int_0^p f(r)e^{-nsp}e^{-sr}dr \\ &= \left(\sum_0^\infty e^{-nsp}\right)\int_0^p f(r)e^{-sr}dr = \frac{1}{1-e^{-ps}}\int_0^p f(r)e^{-sr}dr.\end{aligned}$$

18. $f(t) = 2H(t) - H(t-1) + H(t-2) - H(t-3) + H(t-4) + \cdots$ .

20. Take the $s$-derivative of $F(s)$ by bringing the derivative under the integral sign to get

$$F'(s) = \int_0^\infty f(t)(-te^{-st})dt = \int_0^\infty (-tf(t))e^{-st}dt = -\mathcal{L}(tf(t)).$$

22. Proceed as in Exercise 20 by bringing the s-derivatives under the integral sign to get

$$F''(s) = \frac{d}{ds}F'(s) = \int_0^\infty (-tf(t))(-te^{-st})dt = \int_0^\infty t^2 f(t)e^{-st}dt = \mathcal{L}(t^2 f(t)),$$

and so on. Each derivative increases the power on $t$ by one, and they alternate in sign.

24. Taking transforms of the equations gives $(s-1)X + 2Y = 1,\ -3X + (s-1)Y = 0$. Solving for $X$ and $Y$ gives

$$X(s) = \frac{s-1}{(s-1)^2+6}, \quad Y(s) = \frac{3}{(s-1)^2+6}.$$

Therefore $x(t) = e^t \cos\sqrt{6}\,t,\ \ y(t) = (3/\sqrt{6})e^t\cos\sqrt{6}\,t$.

## ■ **Section 3.3**, **page 162**

2. Take $x(t) = t$ and $y(t) = 1$. Then $\mathcal{L}(t) = \frac{1}{s^2}$; yet $\mathcal{L}(t)\mathcal{L}(1) = \frac{1}{s^2}\frac{1}{s}$.

4. (a) Note that $f(t) = e^t \star t$, so $\mathcal{L}(f) = \mathcal{L}(e^t)\mathcal{L}(t) = \frac{1}{s-1}\frac{1}{s}$.

6. Take the transform of the equation to get $s^2X(s) - \omega^2 X(s) = F(s)$. Thus $X(s) = \frac{1}{s^2-\omega^2}F(s)$. By convolution

$$x(t) = \frac{1}{\omega}\sinh \omega t \star f(t) = \frac{1}{\omega}\int_0^t \sinh \omega(t-\tau)f(\tau)d\tau.$$

8. Taking transforms, $s^2X + 3sX + 2X = \frac{1}{s+4}$. Thus

$$X(s) = \frac{1}{(s+1)(s+2)}\frac{1}{s+4}.$$

Therefore,

$$x(t) = \mathcal{L}^{-1}\left(\frac{1}{(s+1)(s+2)}\right) \star \mathcal{L}^{-1}\left(\frac{1}{s+4}\right).$$

The two inverse transforms on the right are found in the Table.

10. Use the convolution theorem to invert the equation and get

$$x(t) = H(t-3) \star f(t) = \int_0^t H(\tau-3)f(t-\tau)d\tau = \int_3^t f(t-\tau)d\tau.$$

12. $\mathcal{L}^{-1}\frac{1}{s^2(s^2+1)} = t \star \sin t = \int_0^t \tau \sin(t-\tau)d\tau$.

14. Take the transform of the equation using convolution on the integral to get $X(s) = F(s) + K(s)X(s)$. Thus $X(s) = F(s)/(1-K(s))$.

16. The integral term in the equation is the convolution $t^2 \star x(t)$. Thus, taking the transform of the equation gives $sX(s) - \frac{1}{s^3}X(s) = -\frac{1}{s^2}$. Therefore

$$X(s) = -\frac{s}{s^4-1} = -\frac{s}{s^2+1}\frac{1}{s^2-1}.$$

The first factor on the right inverts to $\cos t$ and the second converts to $\sinh t$. Therefore by the convolution theorem, $x(t) = -\int_0^t \cos(t-\tau)\sinh \tau \, d\tau$. The integral can be found by writing cos and sinh in terms of exponential functions.

## ■ **Section 3.4**, **page 173**

2. Taking the transform of the equation and solving for $X(s)$ gives

$$X(s) = \frac{1}{s+3} + \frac{1}{s+3}e^{-s} + \frac{1}{s(s+3)}e^{-4s}.$$

Therefore

$$x(t) = e{-}3t + e{-}3(t-1)H(t-1) - \frac{1}{3}\left(1 - e^{-3(t-4)}\right)H(t-4).$$

4. Taking the transform and solving for $X(s)$ gives $X(s) = \frac{1}{s^2+1}e^{-2s}$. Therefore $x(t) = \sin(t-2)H(t-2)$.

6. Taking the transform and solving for $X(s)$ gives $X(s) = \frac{1}{s^2+1}e^{-2s} - \frac{1}{s^2+1}e^{-5s}$. Thus $x(t) = H(t-2)\sin(t-2) - H(t-5)\sin(t-5)$.

8. The differential equation is $x'' + x = \sum_{n=0}^{\infty} \delta_{n\pi}(t)$ with zero initial conditions. Taking the transform and solving for $X(s)$ gives

$$X(s) = \frac{1}{s^2+1}\sum_{n=0}^{\infty} e^{-n\pi s} = \frac{1}{s^2+1} + \frac{1}{s^2+1}\sum_{n=1}^{\infty} e^{-n\pi s}.$$

Thus

$$x(t) = \sin t + \sum_{n=1}^{\infty} \sin(t-n\pi)H(t-n\pi).$$

10. Solving for $Y(s)$ gives

$$Y(s) = \frac{1}{s^2+s+1}e^{-s} = \sqrt{4/3}\frac{\sqrt{3/4}}{(s+\frac{1}{2})^2 + \sqrt{3/4}^2}e^{-s}.$$

Therefore, from the Table,

$$y(t) = \sqrt{4/3}\,\sin\left(\sqrt{3/4}\,(t-1)\right)e^{-(t-1)/2}.$$

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## Chapter 4 Exercises

### ■ Sec. 4.1, page 190

1. (d) The orbits are on $y = -2x$. As t ranges over $-\infty$ to $\infty$ the orbit goes from infinity to the origin along the straight line $y = -2x$ in the 4th quadrant.

2. (a) The orbits are the ellipses $x^2 + 3y^2/2 = C$. By eliminating y from the differential equations we get $x'' + 6x = 0$, which has solution $x(t) = c_1 \cos\sqrt{6}\,t + c_2 \sin\sqrt{6}\,t$. Then $y(t) = \frac{\sqrt{6}}{3}c_1 \sin\sqrt{6}\,t - \frac{\sqrt{6}}{3}c_2 \cos\sqrt{6}\,t$. So,

$$\mathbf{x}(t) = c_1\begin{pmatrix} \cos\sqrt{6}\,t \\ \frac{\sqrt{6}}{3}\sin\sqrt{6}\,t \end{pmatrix} + c_2\begin{pmatrix} \sin\sqrt{6}\,t \\ \frac{-\sqrt{6}}{3}\cos\sqrt{6}\,t \end{pmatrix}.$$

3. (c) The system is equivalent to $x'' - x' - 2x = 0$ which has $x(t) = c_1 e^{-t} + c_2 e^{2t}$. Then $y(t) = \frac{1}{2}(x' - x) = \frac{1}{2}(-2c_1 e^{-t} + c_2 e^{2t}$. Then

$$\mathbf{x}(t) = c_1\begin{pmatrix} e^{-t} \\ -e^{-t} \end{pmatrix} + c_2\begin{pmatrix} e^{2t} \\ \frac{1}{2}e^{2t} \end{pmatrix}.$$

4. The $y$ equation becomes $y' = \beta x - (\alpha + \gamma)y + k$.

■ **Sec. 4.2.1, page 198**

2. We have $A^{-1} = \begin{pmatrix} -2 & 3/2 \\ 1 & -1/2 \end{pmatrix}$. The solution to the systems is $x = -2.5$, $y = 1.5$. Geometrically, the solution is the intersection of the two straight lines $x + 3y = 2$, $2x + 4y = 1$.

4. For Exercise 3(a) $\lambda = -1, 7$. For Exercise 3(b), when $\lambda = -1$ there are infinitely many solutions lying on the line $3x + 8y = 0$; when $\lambda = 7$ there are infinitely many solutions $x = 0$ (the entire $y$ axis).

6. (a) The determinant is zero; there is no solution unless $m = -5/3$, in which case there are infinitely many solutions lying on the line $-6x - 9y = 5$.

   (b) The determinant is $2 - m$. There is no solution if $m = 2$, and there is a unique solution when $m \neq 2$.

8. We have $\det \begin{pmatrix} 2 & -4 \\ -3 & 8 \end{pmatrix} = 4 \neq 0$. Therefore the vectors are linearly independent.

■ **Sec. 4.2.2, page 201**

2. (a) $(0, 0)$. (b) $(0, 0)$. (e) The line $x - 2y = 0$.

3. (a) The critical point is $x^* = -14$, $y^* = 28/3$. Make the transformation $\mathbf{z} = (x + 14, y - 28/3)^{\mathrm{T}}$ to get the linear system $z_1' = 2z_1 + 3z_2$, $z_2' = -z_1$.

■ **Sec. 4.3, page 206**

2. (a) The characteristic equation is $\lambda^2 - 2\lambda + \beta = 0$. The eigenvalues are $\lambda = 1 \pm \sqrt{1 - \beta}$. If $\beta < 1$ the eigenvalues are real and unequal, if $\beta = 1$ there is a double real eigenvalue, and if $\beta > 1$ the eigenvalues are complex numbers with positive real part.

   (b) The eigenvalues are $\lambda = 2 \pm \sqrt{1 - 2\beta^2}$. If $\beta^2 > \frac{1}{2}$ the eigenvalues are complex; if $\beta^2 < \frac{1}{2}$ the eigenvalues are real and unequal. If $\beta^2 = \frac{1}{2}$ we obtain a double real eigenvalue 2, 2.

4. Assume $A\mathbf{x} = \lambda\mathbf{x}$. Multiply on the left by $A^{-1}$ to get $\mathbf{x} = \lambda A^{-1}\mathbf{x}$, or $A^{-1}\mathbf{x} = \lambda^{-1}\mathbf{x}$.

6. Yes, zero is an eigenvalue for any matrix with zero determinant.

■ **Sec. 4.4.1, page 217**

1. (a) Both eigenvalues are real and negative so the origin is an asymptotically stable node. The general solution is

$$\mathbf{x}(t) = c_1 \begin{pmatrix} -e^{-t} \\ 2e^{-t} \end{pmatrix} + c_2 \begin{pmatrix} e^{-2t} \\ 2e^{-2t} \end{pmatrix}.$$

2. (b) The origin a saddle. The general solution is

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 0 \\ e^{-4t} \end{pmatrix} + c_2 \begin{pmatrix} 5e^t \\ 3e^t \end{pmatrix}.$$

4. $x(t) = 14e^{-2t} - 11e^{-4}$, $y(t) = 21e^{-2t} - 11e^{-4}$.

■ **Sec. 4.4.2**, **page 220**

1. (a) The general solution is

$$\mathbf{x}(t) = c_1 \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + c_2 \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}.$$

2. The eigenvalues are $\lambda = 3/2 \pm i\sqrt{3}/2$, so the origin is an unstable spiral. The vector field is vertical crossing the line $y = x$ and horizontal crossing the line $y = -x/2$. At the point $(0,1)$ we have $x' = -1$, $y' = 2$, which is NW, indicating a counterclockwise orbit.

■ **Sec. 4.4.3**, **page 224**

1. (a) The general solution is

$$x(t) = (c_1 + c_2 t)e^{-3t}, \quad y(t) = c_2 e^{-3t}.$$

■ **Sec. 4.5**, **page 235**

4. (a) The origin is a saddle point.

$$\mathbf{x}(t) = c_1 \begin{pmatrix} -e^{-t} \\ e^{-t} \end{pmatrix} + c_2 \begin{pmatrix} \frac{2}{3}e^{4t} \\ -2e^{4t} \end{pmatrix}.$$

(b) The origin is an asymptotically stable node.

$$x(t) = (c_1 + c_2 t)e^{-3t}, \quad y(t) = c_2 \frac{1}{4} e^{-3t}.$$

(f) The origin is an unstable spiral point.

$$x(t) = -\frac{1}{2}e^t(-c_1(\cos 2t + \sin 2t) + c_2(\cos 2t + \sin 2t))$$
$$y(t) = e^t(c_1 \cos 2t - c_2 \sin 2t).$$

6. There is a line of equilibria $y = x/2$. (b) The eigenpairs are 0, $(2,1)^{\mathrm{T}}$, and 5, $(1,-2)^{\mathrm{T}}$. There is one linear orbit, $\mathbf{x}(t) = (1,-2)^{\mathrm{T}}e^{5t}$. The general solution is

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} e^{5t} \\ -2e^{5t} \end{pmatrix}.$$

The phase diagram shows the line of equilibria with parallel lines coming out of the equilibria of slope $-2$.

8. (a) The characteristic equation is $\lambda^2 + 5\alpha - 4 = 0$. If $5\alpha - 4 > 0$ the eigenvalues are purely imaginary and the origin is a center. For $5\alpha - 4 < 0$ the eigenvalues are real and unequal and the origin is a saddle point. Thus there is a bifurcation at $\alpha = 4/5$.

10. The trace is $T = a$ and the determinant is $D = -2 + a^2/4$. Plotting $D$ vs $T$ gives a parabola in the $TD$ plane which is concave up, and passes through $(\pm\sqrt{8}, 0)$ and $(0, -2)$. Therefore, for $a < -\sqrt{8}$ we have a stable node, for $-\sqrt{8} < a < \sqrt{8}$ a saddle, and for $a > -\sqrt{8}$ an unstable node. Thus, bifurcations occur at $a = \pm\sqrt{8}$.

12. No. The trace of the matrix is nonzero.

14. (d) There are no equilibria. The general solution is

$$x(t) = -c_1 - \frac{1}{3}c_2 e^{-2t} - \frac{3}{4} + \frac{7}{2}t, \quad y(t) = c_1 + c_2 e^{-2t} + \frac{9}{4} - \frac{7}{2}t.$$

16. The origin is a stable spiral point, so the current and voltage go to zero. The general solution is

$$I(t) = -\frac{\sqrt{2}}{2}(c_1 \cos(\sqrt{2}t) + c_2 \sin(\sqrt{2}t))e^{-t}$$
$$V(t) = (-c_1 \sin(\sqrt{2}t) + c_2 \cos(\sqrt{2}t))e^{-t}$$

## ■ Sec. 4.6, page 243

2. The system is

$$(V_1 C_1') = (q + r)c - rC_1 - qC_1, \quad (V_2 C_2)' = qC_1 - qC_2.$$

Initially, $C_1 = C_2 = 0$ at $t = 0$.

4. The fundamental matrix and its inverse are

$$\Phi = \begin{pmatrix} 3e^{-4t} & -e^{-11t} \\ e^{-4t} & 2e^{-11t} \end{pmatrix}, \quad \Phi^{-1} = \frac{1}{7}\begin{pmatrix} 2e^{4t} & e^{4t} \\ -e^{11t} & 3e^{11t} \end{pmatrix}.$$

For undetermined coefficients, take $\mathbf{x}_p(t) = (Ae^{-t}, Be^{-t})^{\mathrm{T}}$. Substitute into the system to get $A = 3/10,\ B = 1/15$.

8. The only equilibrium is (0,0). The $x$ nullcline $y = (r_1 + r_3)/r_2$ lies above the $y$ nullcline $y = r_1/r_2$. In the region under the $y$ nullcline the vector field is NW, between the nullclines it is SW, and above the $x$ nullcline it is SE. This clearly shows the origin is an asymptotically stable node. (Analytically one can show both eigenvalues are negative.) Therefore the solution beginning at $(x_0, 0)$ approaches zero as $t \to \infty$.

10. The model equations are

$$E' = -hE + bP, \quad P' = hE - mP.$$

If $A$ is the coefficient matrix then $\text{tr}A = -h-m < 0$ and $\det A = h(m-b)$. If $m > b$ the $\det A > 0$ and the origin is asymptotically stable; thus the population dies out. If $m < b$ then $\det A < 0$ and the origin is a saddle point. The population grows without bound. If eggs are eaten at the constant rate $\rho$, then the equations become

$$E' = -hE + bP - \rho, \quad P' = hE - mP.$$

The equilibrium is now

$$E = \frac{\rho m}{h(b-m)} \quad P = \frac{\rho}{b-m},$$

which is viable only if $b > m$. In this case the population approaches the coexistent equilibrium state. In this case the equilibrium is a saddle and, depending on the initial conditions, orbits go to infinity or go to the line $E = 0$ (no eggs). If $b < m$ there are no equilibria and the egg population dies out.

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## Chapter 5 Exercises

### ■ Sec. 5.1, page 257

2. Note that the Jacobian matrix is $J(x,y) = \begin{pmatrix} 1-y+2hx & -x \\ y & x-1 \end{pmatrix}$. In all cases, regardless of the value of $h$, the Jacobian matrix shows that the origin $(0,0)$ is a saddle point. The two other critical points are $(-1/h, 0)$ and $(1, h+1)$. The Jacobian matrix at those two critical points in the cases $h = 1, -1, 8$ give the results, which we leave to the reader. The nullclines show that orbits are vertical along $x = 0$ and $y = 1 + hx$ and horizontal along $y = 0$ and $x = 1$.

4. The $x$ nullcline is the circle $x^2 + y^2 = 4$ of radius 2; the $y$ nullcline is the straight line $y = 2x$. These intersect at the two equilibria $(2/\sqrt{5}, 4/\sqrt{5}$ and $(-2/\sqrt{5}, -4/\sqrt{5}$. To get the direction field note $x' > 0$ outside the circle and $x' < 0$ inside the circle; $y' > 0$ above the straight line and $y' < 0$ below the straight line. This shows $(-2/\sqrt{5}, -4/\sqrt{5}$ is a saddle point. The orbits near $(2/\sqrt{5}, 4/\sqrt{5}$ have a circular pattern, so it is not clear if that critical point is a center or a spiral.

6. Setting $\sin y = 0$ and $x = 0$ gives equilibria $(0, n\pi)$, $n = 0, \pm 1, \pm 2, \ldots$, all along the $y$ axis.

8. Dividing the two equations and separating variables gives $(1/y)dy = e^x/(e^x - 1)$. Integrating yields the orbits $y = C(e^x - 1)$. The only critical point is $(0,0)$ and examining the Jacobian matrix shows it is an unstable node with eigenvalues $1, 1$.

10. By checking the Jacobian matrix, the critical points are $(0,0)$ (saddle point) and $(2,0)$ (asymptotically stable node).

12. The critical points are $(1,0)$ (saddle point) and $(-1,0)$ (stable spiral). The orbits are vertical along $y = 0$ and horizontal along the parabola $y = x^2 - 1$.

14. The equilibrium is (0,0) and the Jacobian matrix is $J(0,0) = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$.
The trace is $-1$ (negative) and the determinant is $+1$ (positive), so by the theorem $(0,0)$ is asymptotically stable.

## ■ Sec. 5.2, page 269

2. $V(x) = -\int F(x)dx = \frac{1}{2}x^2 - \frac{1}{4}x^4$. Conservation of energy is

$$\frac{1}{2}y^2 + \frac{1}{2}x^2 - \frac{1}{4}x^4 = E, \quad x' = y.$$

At $x = 2$, $x' = 1$ when $t = 0$ we get $E = -3/2$. The orbit is given by

$$y = \pm\sqrt{2}\sqrt{-3/2 - x^2/2 + x^4/4}.$$

4. $V(x) = -\int F(x)dx = x^3/3$. Conservation of energy is $\frac{1}{2}y^2 + \frac{1}{3}x^3 = \frac{1}{3}$, where we used the initial conditions to obtain $E = 1/3$. Solving for $y$ gives the orbit $y = \pm\sqrt{2/3}\sqrt{1 - x^3}$. Replacing $y$ by $dx/dt$, separating variables, and then integrating gives the implicit solution

$$\pm \int \frac{1}{1 - x^3} dx = \sqrt{2/3}\, t + C.$$

6. The force is $F(x) = -V'(x) = -2(x+1)(x-2)(2x-1)$. Conservation of energy is $y^2 + (x+1)^2(x-2)^2 = E$, giving orbits $y = \pm\sqrt{E - (x+1)^2(x-2)^2}$. Not that orbits move to the right in the upper half plane and to the left in the lower half plane. Specifically, when $x = 0$, $y = 3$ initially, we find $E = \sqrt{13}$.

8. The conservation law is given to be $\frac{1}{2}ml^2(\theta')^2 + mgl(1 - cos\theta) = E$. Taking the derivative with respect to $t$, gives

$$\frac{1}{2}ml^2 2\theta'\theta'' + mgl\sin(\theta)\theta' = 0.$$

This immediately reduces to $\theta'' + (g/l)\sin\theta = 0$.

10. Substituting the given quantities into the conservation law gives $E = 1.699(10)^{-3}$ J.

12. The critical points are $(1, 0)$ and $(-1, 0)$. The Jacobian matrix shows easily that $(1, 0)$ is a saddle point and $(-1, 0)$ is an asymptotically stable spiral.

14. We have $E = \frac{1}{2}m(x')^2 + V(x)$. Then, taking the time derivative,

$$\begin{aligned} \frac{dE}{dt} &= mx'x'' + V'(x)x' \\ &= x'(mx'' + V'(x)) \\ &= x'(-kx') \qquad \text{(by the equation of motion)} \\ &= -k(x')^2 < 0. \end{aligned}$$

## ■ Sec. 5.3.2, page 282

4. The $x$ nullclines are the straight lines $x = 0$, $y = a/b$. The $y$ nullcline is the curve $y = M/(dx - c)$ which has a vertical asymptote at $x = c/d$; that is, $y \to +\infty$ as $x \to c/d+$, and $y \to -\infty$ as $x \to c/d-$. There is one equilibrium at $((Mb + ac)/ad, a/b)$. The trace of the Jacobian matrix at the equilibrium is $Mb/a$, which is positive. Therefore, it is unstable. A direction field shows a counterclockwise circulation, so the equilibrium is an unstable spiral. Eventually the predator population will become extinct as an orbit intersects the $x$ axis.

6. The equilibria are at $(0, 0)$ and $(4, 6)$. The Jacobian matrix shows $(0, 0)$ is an unstable node and $(4, 6)$ is an asymptotically stable node. The latter is a coexistent state in this competition problem. Regardless of the initial condition, the orbit approaches $(4, 6)$.

10. The nullclines $y = x$ and $y = 5x^2/(4 + x^2)$ (a sigmoid type curve) cross at the critical points $(0, 0)$, $(1, 1)$, $(4, 4)$. The Jacobian matrix is $J(x, y) = \begin{pmatrix} -1 & 1 \\ \frac{40x}{(4+x^2)^2} & -1 \end{pmatrix}$. Evaluating $J$ at the critical points shows directly that $(0, 0)$ is an asymptotically stable node, $(1, 1)$ is a saddle point, and $(4, 4)$ is an asymptotically stable node.

14. (a) Because $f(0) = 0$, substituting $(0, 0)$ into the equations shows $x' = 0$, $y' = 0$. So $(0, 0)$ is an equilibrium. When $x = K$ and $y = 0$ again $x' = 0$, $y' = 0$. So $(K, 0)$ is an equilibrium. (b) The Jacobian at (0,0) is

$$J(x, y) = \begin{pmatrix} r - 2rx/K & -f(x) \\ cf'(x)y & -m + cf(x) \end{pmatrix}_{(0,0)} = \begin{pmatrix} r & 0 \\ 0 & -m \end{pmatrix}.$$

The eigenvalues are real and opposite sign, and thus $(0, 0)$ is a saddle point. Evaluating the Jacobian at $(K, 0)$ gives

$$J(K, 0) = \begin{pmatrix} -r & -f(K) \\ 0 & -m + cf(K) \end{pmatrix}.$$

The eigenvalues are $\lambda_1 = -r$ and $\lambda_2 = -m + cf(K)$. If $m > cf(K)$ then both eigenvalues are negative and $(K, 0)$ is an asymptotically stable node; if $m < cf(K)$ then eigenvalues have opposite signs and $(K, 0)$ is a saddle point. (c) For a positive equilibrium we must have $y' = y(-m+cf(x)) = 0$; this means there must be a positive solution $x^*$ to $f(x) = m/c$, or $f(x^*) = m/c$. Because $f(x) \to M$ as $x \to +\infty$, we need $M > m/c$. Therefore, to get $x' = 0$ we need $rx^*(1 - x^*/K) - f(x^*)y = 0$ or $y = \frac{mr}{c}x^*(1 - x^*/K)$. This must be positive, so $x^* < K$.

■ **Sec. 5.3.3**, **page 292**

4. Here $r = 1/3$, $S(0) = 180$, $N = 200$, and $S^* = 100$ did not get the flu. Therefore the equation $-S^* + N + \frac{1}{3a}\ln(S^*/S_0) = 0$ holds. Substituting and solving for $a$ gives $a = 0.001959$. The maximum number of infectives occur when $S = r/a = 170$. Substituting values into $I = -S + N + \frac{r}{a}\ln(S/S_0)$ gives $I = 20$.

8. We need only the first two equations since $R = N - S - I$ will then be determined. In the SI plane only the equilibrium point is (0,0). The Jacobian matrix at that point has two negative eigenvalues and thus is an asymptotically stable node. Eventually, therefore, there are no infected and susceptible individuals remaining and all become removed. To get the phase plane, note that $S' < 0$, always, and $I' > 0$ if $S > r/a$, $I' < 0 if S < r/a$. Therefore, if $S(0) > r/a$, and $I(0)$ is small, the orbit rises as infectives increase, it crosses the line $S = r/a$ horizontally, and then goes to $(0, 0)$ as $t \to \infty$.

10. (a) The reaction rate is $r = kx$ and $x' = -kx$ and $y' = 2kx$. Note that $2x + y = C = 2x_0 + y_0$. Therefore $y = 2x_0 + y_0 - 2x$. But $x = x_0e^{-kt}$, and $y$ is given by the last equation. (b) There are two reactions with rates $r_1 = k_1ax$, $r_2 = k_{-1}x^2$. We have $x' = k_1ax - k_{-1}x^2$, which is a logistic equation with unstable equilibrium $x = 0$ and stable equilibrium $x = k_1a/k_{-1}$. (d) The rate is $r = kx^2y$. Therefore, $x' = 3kx^2y$, $y' = -kx^2y$, $z' = kx^2y$. Note that $x+3y = C$, where $C = x_0+3y_0$. Therefore $y = (1/3)(C-x)$, and the $x$ equation becomes $x' = kx^2(C-x)$. The phase line diagram shows $x = C$ is a stable equilibrium, so $x \to C$. Thus, $y \to 0$; and because $y+z = C_1 = y_0$ we get $z = y_0 - y \to y_0$. (f) The reactions have rates $r_1 = k_1x$, $r_2 = k_2y$. Therefore the equations are $x' = -k_1x + k_2y$, $y' = k_1x - k_2$, $z' = k_2y$. Clearly $x + y = C = x_0$. Thus, $x' = -(k_1 + k_2)x + k_2x_0$. This has a stable equilibrium at $x^* = k_2x_0/(k_1 + k_2)$. Therefore, $y \to x_0 - x^*$.

12. First show that $(S+I+I)' = 0$ by adding the equations; thus $S+E+R = N$. Making the change of variables as indicated, we get

$$x' = \mu(1 - x) - xy, \quad y' = (r + \mu)y(R_0x - 1), \quad R_0 > 1.$$

The Jacobian at the equilibrium $(1, 0)$ has two negative eigenvalues and is therefore a saddle. The other equilibrium $(R_0^{-1}, (r + \mu)(R_0 - 1))$ is an asymptotically stable endemic state. Check the sign of $(\text{tr}\, J)^2 - 4\det J$ to determine when the critical point is a node or spiral. (Hint: Look at the cases when $R_0$ is slightly greater than 1 and when $R_0$ is well above 1.)

■ **Sec. 5.4**, **page 306**

2. Because $x' > 0$, there can never be a critical point. So there are no periodic orbits.

4. Note if $k = 0$ we can take $V'(x) = x^2$ to get $x'' + 2x = 0$, which has periodic orbits. If $k \neq 0$ then the Dulac theorem gives $f_x + g_y = 0 + (-k) < 0$, which is of one sign. Therefore there are no periodic orbits.

6. Hint: Use the Poincare–Bendixson theorem.

8. Clearly $\theta(t) = -t + c_1$, and the polar angle rotates clockwise. The equation $r' = r(1 - r^2)$ is a Bernoulli equation. Make the substitution $w = r^{-2}$ to get $w' + 2w = 2$. So $w = 1 + c_2 e^{-2t}$, giving

$$r(t) = \frac{1}{\sqrt{1 + c_2 e^{-2t}}}.$$

Note $c_2 = 0$ gives the circular orbit $r = 1$; as $t \to \infty$ note $r \to 1$ from outside the circle or from inside the circle, depending on $c_2 > 0$ or $c_2 < 0$, respectively.

10. We have $f = H_y = 8y$, $g = -H_x = -2x$. So the system is $x' = 8y$, $y' = -2x$. The origin $(0,0)$ is the only equilibrium and by dividing the differential equations, separating variables, and integrating we get orbits $0.5x^2 + 2y^2 = C$, which are circles. The origin is a center.

12. To get part (a) use the law of mass action. (b) Note that $(x - 2y - z)' = x' - 2y' - z' = 0$ (by substituting the equations) and so $x - 2y - z = C$. (c) Note $z(0) = 0$ so $z = x - 2y$. Substituting into the two equations we can eliminate $z$ to get $x' = (2\beta - \alpha)xy - \beta x^2$ and $y' = -\alpha xy$. (d) There is a line of critical points, $x = 0$ or the $y$ axis. The $x$ nullcline is along the straight line $y = \beta x/(2\beta - \alpha)$. Above the nullcline the vector field is SE and below the nullcline it is SW. So orbits approach the origin $x = 0$, $y = 0$.

■ **Sec. 5.5**, **page 311**

2. The equilibria are (0,0) and (1,0). The Jacobian matrices at the critical points are

$$J(0,0) = \begin{pmatrix} 0 & 1 \\ -1 & c \end{pmatrix} \qquad J(1,0) = \begin{pmatrix} 0 & 1 \\ 1 & c \end{pmatrix}.$$

For $J(0,0)$ the trace and determinant are positive. So $(0,0)$ is unstable; it is an unstable spiral point if $c < 2$ and an unstable node if $c > 2$ (because the discriminant is $c^2 - 4$.). For $J(1,0)$ we have the determinant negative, so $(1,0)$ is always a saddle point. So the bifurcation occurs at $c = 2$. As $c$ increases from 0, the origin changes from to an unstable spiral to an unstable node at $c = 2$.

4. This problem can be handled exactly like Exercise 8, Section 5.4. Alternately, we first note $r = \sqrt{a}$ is a circular, periodic solution. If $a > 0$ then the only equilibrium is $(0,0)$. Further, $\theta(t) = -t + C$ winds clockwise. The Jacobian matrix of the original rectangular system is

$$J(0,0) = \begin{pmatrix} a & 1 \\ -1 & a \end{pmatrix}.$$

Note that $\operatorname{tr}(J) = 2a$ and $\det J = a^2 + 1 > 0$. The discriminant is negative. If $a < 0$ the trace is negative and determinant is positive, so $(0,0)$ is a stable spiral. If $a > 0$ the trace is positive and $(0,0)$ is an unstable spiral; clearly, from the $r$-equation the orbits approach the limit cycle $r = \sqrt{a}$.

6. For $h < 0$ there are no critical points; if $h = 0$ the only critical point is $(0,0)$; for $h > 0$ there are two critical points $(\sqrt{h}, 0)$ and $(-\sqrt{h}, 0)$. For $h = 0$ the origin has a nonstandard orbital structure with a saddle structure for $x > 0$ and a stable node structure for $x < 0$. The Jacobian matrices at the two critical points are

$$J(\sqrt{h}, 0) = \begin{pmatrix} 2\sqrt{h} & 0 \\ 0 & -1 \end{pmatrix} \qquad J(-\sqrt{h}, 0) = \begin{pmatrix} -2\sqrt{h} & 0 \\ 0 & -1 \end{pmatrix}.$$

These give a saddle and a stable node, respectively.

8. For $a < 0$ there are no critical points. For $a > 0$ there are two critical points $(\sqrt{a}/2, 0)$ and $(-\sqrt{a}/2, 0)$. It is easily seen by calculating the Jacobian matrices at these two points that they are a saddle point and an asymptotically stable node, respectively. Therefore, as a passes through zero two equilibria appear at $a = 0$.

10. The only critical point is $(1, a)$. The Jacobian matrix is

$$J(1,a) = \begin{pmatrix} a-1 & -1 \\ 3a & -1 \end{pmatrix}.$$

If $a < 2$ the trace is negative and determinant positive, thus $(1,a)$ is asymptotically stable. If $a > 2$ the trace and determinant are positive, so $(1,a)$ is unstable.

12. (a) Kirchhoff's law states $V_L + V_R + V_C = 0$ or $LI' + f(I) + \frac{1}{C}Q = 0$. Taking the derivative while using the chain rule gives $LI'' + f'(I)I' = I = 0$. (b) Direct substitution into the last equation. (d) The system is $x' = y,\ \ y' = -x - \mu(x^2 - 1)y$. The Jacobian matrix at $(0,0)$ is

$$J(0,0) = \begin{pmatrix} 0 & 1 \\ -1 & \mu \end{pmatrix}.$$

The trace is $\mu$ and determinant is 1. The discriminant is $\mu^2 - 4$. Therefore, if $\mu > 2$ the origin is an unstable node, and if $\mu < 2$ it is an unstable spiral.

14. This is a linear system with critical point $(0, 0)$. The Jacobian matrix is

$$J(0,0) = \begin{pmatrix} -(\nu + \lambda N) & b \\ \lambda N & -\mu \end{pmatrix}.$$

The trace is clearly negative. The determinant is $\det J = \mu(\nu+\lambda N) - b\lambda N$. If $\det J > 0$ the the origin is asymptotically stable and the populations die out. We want the other case when an outbreak occurs, namely when the determinant is positive, so the origin is unstable. This occurs when $\mu(\nu + \lambda N) < b\lambda N$, which we now assume. In this case the $L$ nullcline lies below the $M$ nullcline in the first quadrant (by our assumed determinant condition) and so $(0, 0)$ is a saddle point; thus orbits must veer away from the origin and go to infinity.

18. Take the square with vertices at $(0, \pm 1)$, $(\pm 1, 0)$ and find the vector field along its edges.

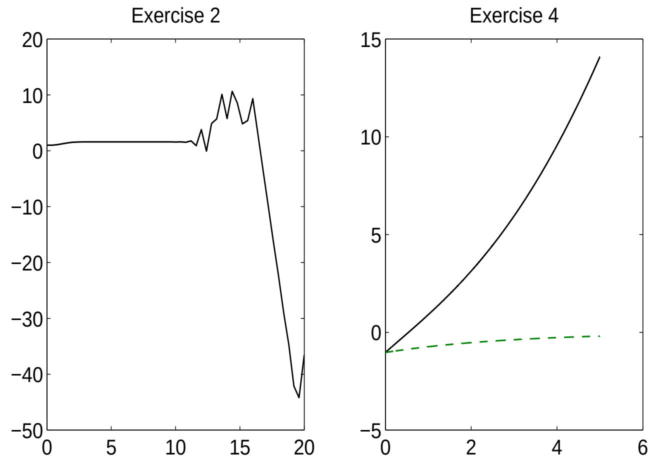


Figure 5: (Left) Euler approximation in Exercise 2 with $N = 50$. (Right) The Euler approximation with $h = 0.001$ in Exercise 4 along with the exact solution (dashed).

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## Chapter 6 Exercises

■ **Sec. 6.1, page 320**

2. The recursion formula is $x_{k+1}(t) = 1 + \int_0^t (s - x_k(s))ds$. Then $x_0(t) = 1$, and $x_1(t) = 1 + \int_0^t (s - x_0(s))ds = 1 + \int_0^t (s-1)ds = 1 + (1/2)t^2 - t$, etc.

■ **Sec. 6.3, page 330**

2. A plot of the solution using the Euler method with $N = 50$ and $h = 0.4$ is shown in Figure 5, left panel. The approximation is clearly inaccurate; the exact solution is $x_{\text{ex}}(t) = \exp(\sin t)$, which is periodic. We leave it to the reader to check the additional cases for possible improvement.

4. A plot of the solution using the Euler method with $N = 5000$ and $h = 0.001$ is shown in Figure 5, right panel, along with the exact solution

$$x_{\text{ex}}(t) = -\frac{33}{32}e^t e^{-4t/3}.$$

   The slope field is shown in Figure 6, indicating the reason for the inaccuracy.

6. The answer is similar to Exercise 4. The solution is relatively flat near the exact solution. But if the initial condition is changed slightly, the solution

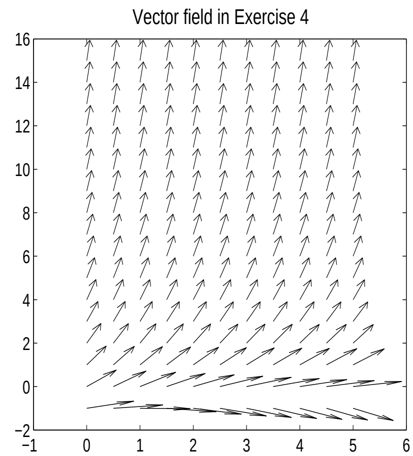


Figure 6: The slope field in Exercise 4 showing the flatness in the region near the solution. But there is a dramatic steepness in the slope field near that region. A small error in the Euler method, which follows the slope field, leads to a completely inaccurate approximation.

curves blow up. In the Euler method the approximation moves away from the exact solution and is taken to the region where the slope field is steep. This phenomenon can be compared to attempting to approximate a separatrix moving into an unstable saddle point of a system; nearby orbits veer away from the saddle near the critical point.

8 . The differential equation is $x'(t) = f(t)$. Integrating from $t_n$ to $t_{n+1}$ and using the fundamental theorem of calculus, we get

$$x(t_{n+1}) = \int_{t_n}^{t_{n+1}} f(t)dt.$$

Approximating the right side by Simpson's rule (see any calculus book) gives

$$\begin{aligned}\int_{t_n}^{t_{n+1}} f(t)dt &= \frac{t_{n+1}-t_n}{6}\left[f(t_n) + 4f\left(\frac{t_{n+1}+t_n}{2}\right) + f(t_{n+1})\right] + \mathrm{O}(h^5) \\ &= \frac{h}{6}(f(t_n) + 4f(t_n + h/2) + f(t_{n+1})) + \mathrm{O}(h^5).\end{aligned}$$

This is the Runge–Kutta formula with $k_1 = f(t_n)$, $k_2 = k_3 = f(t_n + h/2)$, $k_4 = f(t_{n+1})$.