1 Matrices

The study of linear systems is facilitated by introducing matrices. Matrix theory provides a convenient language and notation to express many of the ideas concisely, and complicated formulas are simplified considerably in this framework. And, importantly, matrix notation is more or less independent of dimension. Therefore the results we present for two equations in two unknowns will extend easily to n equations in n unknowns. In the first section we present a brief introduction to square matrices Some of the definitions and properties will be given for general n by n matrices, but our focus is mostly on the two-dimensional case.

A square array A of numbers having n rows and n columns is called a **square matrix** of size n, or an $n \times n$ matrix. The number in the ith row and jth column is denoted by a_{ij}. A general 2×2 matrix has the form

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

The numbers a_{ij} are called the **entries** of the matrix; the first subscript i denotes the row, and the second subscript j denotes the column. We often denote matrices using the brief notation $A = (a_{ij})$. An n-**vector x** is a list of n numbers $x_1, x_2, ..., x_n$, written as a *column*; so "vector" means "column list". The numbers $x_1, x_2, ..., x_n$ in the list are called its **components**. For example,

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

is a 2-vector. Vectors will be denoted by lower case boldface letters like **x**, **y**, etc., and matrices will be denoted by capital letters like A, B, etc. To minimize space in type setting, we often write, for example, a 2-vector **x** as $(x_1, x_2)^{\mathrm{T}}$, where the T denotes "transpose", meaning turn the row into a column.

Two square matrices having the same size can be added entrywise. That is, if $A = (a_{ij})$ and $B = (b_{ij})$ are both $n \times n$ matrices, then the **sum** $A + B$ is an $n \times n$ matrix defined by $A + B = (a_{ij} + b_{ij})$. A square matrix $A = (a_{ij})$ of any size can be multiplied by a constant c by multiplying all the elements of A by the constant; in symbols this **scalar multiplication** is defined by $cA = (ca_{ij})$. Thus $-A = (-a_{ij})$, and it is clear that $A + (-A) = 0$, where 0 is the **zero matrix** having all entries zero. If A and B have the same size, then **subtraction** is defined by $A - B = A + (-B)$. Also, $A + 0 = A$, if 0 has the same size as A. Addition, when defined, is both commutative and associative. Therefore the arithmetic rules of addition for n by n matrices are the same as the usual rules of algebra of real numbers.

Similar rules hold for addition of column vectors of the same size and multiplication of column vectors by scalars; these are the definitions encountered in multivariable calculus: vectors add componentwise, and multiplication of a vector by a scalar multiplies each component of that vector by the scalar.

Example 5.3 Let

$$A = \begin{pmatrix} 1 & 2 \\ 3 & -4 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -2 \\ 7 & -4 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} -4 \\ 6 \end{pmatrix} \quad \mathbf{y} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}.$$

Then

$$\begin{aligned} A + B &= \begin{pmatrix} 1 & 0 \\ 10 & -8 \end{pmatrix}, \quad -3B = \begin{pmatrix} 0 & 6 \\ -21 & 12 \end{pmatrix}, \\ A^T &= \begin{pmatrix} 1 & 3 \\ 2 & -4 \end{pmatrix}, \quad 5\mathbf{x} = \begin{pmatrix} -20 \\ 30 \end{pmatrix}, \quad \mathbf{x} + \mathbf{y} = \begin{pmatrix} 1 \\ 7 \end{pmatrix}. \end{aligned}$$

The product of two square matrices of the same size is *not* found by multiplying entrywise. Rather, **matrix multiplication** is defined as follows. Let A be and n by n matrix and B be an n by n matrix. Then the matrix AB is defined to be the n by n matrix $C = (c_{ij})$ where the ij entry (in the ith row and jth column) of the product C is found by taking the product (dot product, as with vectors) of the ith row of A and the jth column of B. In symbols, $AB = C$, where

$$c_{ij} = \mathbf{a}_i \cdot \mathbf{b}_j = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj},$$

where $\mathbf{a}_i$ dentoes the ith row of A, and $\mathbf{b}_j$ denotes the jth column of B.. Generally, matrix multiplication is *not* commutative, i.e., $AB \neq AB$, so the order in which matrices are multiplied is important. However, the associative law $AB(C) = (AB)C$ does hold, so you can regroup products as you wish.The distributive law connecting addition and multiplication, $A(B + C) = AB + AC$, also holds. The powers of a square matrix are defined by $A^2 = AA$, $A^3 = AA^2$, and so on.

Example 5.4 Let

$$A = \begin{pmatrix} 2 & 3 \\ -1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 4 \\ 5 & 2 \end{pmatrix}.$$

Then

$$AB = \begin{pmatrix} 2 \cdot 1 + 3 \cdot 5 & 2 \cdot 4 + 3 \cdot 2 \\ -1 \cdot 1 + 0 \cdot 5 & -1 \cdot 4 + 0 \cdot 2 \end{pmatrix} = \begin{pmatrix} 10 & 14 \\ -1 & -4 \end{pmatrix}.$$

Also

$$\begin{aligned} A^2 &= \begin{pmatrix} 2 & 3 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ -1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 2 \cdot 2 + 3 \cdot (-1) & 2 \cdot 3 + 3 \cdot 0 \\ -1 \cdot 2 + 0 \cdot (-1) & -1 \cdot 3 + 0 \cdot 0 \end{pmatrix} = \begin{pmatrix} -1 & 6 \\ -2 & -3 \end{pmatrix}. \end{aligned}$$

Next we define multiplication of a $n \times n$ matrix A times an n-vector $\mathbf{x}$. The product $A\mathbf{x}$ is defined to be an n-vector whose ith component is $\mathbf{a}_i \cdot \mathbf{x}$.In other words, the ith element in the column vector $A\mathbf{x}$ is found by taking the product (dot product) of the ith row of A and the vector $\mathbf{x}$. The product $\mathbf{x}A$ is not defined.

Example 5.5 When $n = 2$ we have

$$A\mathbf{x} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}.$$

As a numerical example take

$$A = \begin{pmatrix} 2 & 3 \\ -1 & 0 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} 5 \\ 7 \end{pmatrix}.$$

Then

$$A\mathbf{x} = \begin{pmatrix} 2 \cdot 5 + 3 \cdot 7 \\ -1 \cdot 5 + 0 \cdot 7 \end{pmatrix} = \begin{pmatrix} 31 \\ -5 \end{pmatrix}.$$

The **main diagonal** of an $n \times n$ matrix A are the entries a_{ii}, $i = 1, 2, ..., n$, on the diagonal. A square matrix having ones on the main diagonal and zeros elsewhere is called the **identity matrix** and is denoted by I. . For example, the 2×2 identity is

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

It is not difficult to show that if A is any square matrix, then $AI = IA = A$. Therefore multiplication by the identitiy matrix does not change the result, a situation similar to multiplying real numbers by the unit number 1. If A is an $n \times n$ matrix and there exists a matrix B for which $AB = BA = I$, then B is called the **inverse** of A and we denote it by $B = A^{-1}$. If A^{-1} exists, we say A is a **nonsingular** matrix; otherwise it is called **singular**. One can show that the inverse of a matrix is unique.

Now we indicate how to compute the inverse of a 2×2 matrix. First we define another useful number associated with a square matrix A called its determinant. The **determinant** of a square matrix A, denoted by $\det A$, is a number found by combining the elements of the matrix is a special way. For a 2×2 matrix

$$\det A = \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - cb.$$

Example 5.6 We have

$$\det \begin{pmatrix} 2 & 6 \\ -2 & 0 \end{pmatrix} = 2 \cdot 0 - (-2) \cdot 6 = 12.$$

Now we can give a simple formula for the inverse of a 2×2 matrix A. Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

be a nonsingular matrix. Then

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}. \tag{1}$$

So the inverse of a 2×2 matrix is found by interchanging the main diagonal elements, putting a minus signs on the off-diagonal elements, and dividing by the determinant.

Calculating the determinant of a 3×3 or larger matrix is more computationally involved. There is a simple scheme for a 3×3 matrix that many learned in high school, along with mnemonic devices to help. The formula is

$$\det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = aei + bfg + cdh - cef - bdi - ahf.$$

For larger sized determinants there are no simple schemes. Methods based on row reduction are used in practice, as well as expansion by minors.These methods are encountered in elementary courses in matrix algebra. Computer algebra packages have commands that return the determinant instantly.

Example 5.7 If

$$A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix},$$

then

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} 3 & -2 \\ -4 & 1 \end{pmatrix} = \frac{1}{-3} \begin{pmatrix} 3 & -2 \\ -4 & 1 \end{pmatrix} = \begin{pmatrix} -1 & \frac{2}{3} \\ \frac{4}{3} & -\frac{1}{3} \end{pmatrix}.$$

The reader should check that $AA^{-1} = I$.

Equation (1) is revealing because it seems to indicate the inverse matrix exists only when the determinant is nonzero (you can't divide by zero). In fact, these two statements are equivalent for any square matrix, regardless of the size: A^{-1} exists if, and only if, $\det A \neq 0$. This is a major theoretical result in matrix theory, and it is a convenient test for invertibility for matrices. The reader should remember the equivalences

$$A^{-1}\text{exists} \Leftrightarrow A \text{ is nonsingular} \Leftrightarrow \det A \neq 0.$$

Matrices were developed to represent and study linear algebraic systems (n linear algebraic equations in n unknowns) in a concise way. For example, consider two equations in two unknowns x_1, x_2 given by

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2 \end{aligned}$$

Using matrix notation we can write this simply as

$$A\mathbf{x} = \mathbf{b}, \tag{2}$$

where

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}.$$

A is the **coefficient matrix**, $\mathbf{x}$ is the column vector of unknowns, and $\mathbf{b}$ is the column vector representing the right sides. If $\mathbf{b} = \mathbf{0}$, the zero vector, then the system (2) is called **homogeneous**. Geometrically, each equation in the system represents a line in the plane. When $\mathbf{b} = \mathbf{0}$ the two lines pass through the origin. A solution vector $\mathbf{x}$ is represented by a point that lies on both lines. There can be a unique solution where both lines intersect at a single point; there can be infinitely many solutions where both lines coincide; or there can be no solution. if the lines are parallel.

The following theorem tells us when the linear system (2) is solvable. It is an important result that will be applied often in the sequel.

Theorem 1 *Let A be a 2×2 matrix. If A is nonsingular, then the system $A\mathbf{x} = \mathbf{b}$ has a unique solution given by $\mathbf{x} = A^{-1}\mathbf{b}$; in particular, the homogeneous system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution $\mathbf{x} = \mathbf{0}$. If A is singular, then the homogeneous system $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions, and the nonhomogeneous system $A\mathbf{x} = \mathbf{b}$ may have no solution or infinitely many solutions.*

It is easy to show the first part of the theorem, when A is nonsingular, using the power of matrix notation. If A is nonsingular then A^{-1} exists. Multiplying both sides of $A\mathbf{x} = \mathbf{b}$ by A^{-1}, we get

$$\begin{aligned} A^{-1}A\mathbf{x} &= A^{-1}\mathbf{b}, \\ I\mathbf{x} &= A^{-1}\mathbf{b}, \\ \mathbf{x} &= A^{-1}\mathbf{b}, \end{aligned}$$

which is the unique solution. But, we remark that this method of finding and multiplying by the inverse of the matrix A is not the most efficient method for solving linear systems. The method of Gaussian elimination, introduced in linear algebra, is an efficient algorithm for solving large systems. To prove the theorem in the case that A is singular we appeal to the geometric interpretation. If A is singular, then $\det A = 0$, and the two lines represented by the two equations must be parallel (can you show that?). Therefore they either coincide or they do not, giving either infinitely many solutions or no solution.

It is important to remark that Theorem 1 gives a characterization of solutions of linear systems of any size, not just two equations in two unknowns, but rather n equations in n unknowns.

Example 5.8 Consider the homogeneous linear system

$$\begin{pmatrix} 4 & 1 \\ 8 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The coefficient matrix has determinant zero, so there will be infinitely many solutions. The two equations represented by the system are

$$4x_1 + x_2 = 0, \quad 8x_1 + 2x_2 = 0,$$

which are clearly dependent; one is a multiple of the other. Therefore we need only consider one of the equations, say $4x_1 + x_2 = 0$. This is one equation in two unknowns, so we are free to pick a value for one of the variables and solve for the other one. Let $x_1 = 1$; then $x_2 = -4$ and we get a single solution $\mathbf{x} = (1, -4)^{\mathrm{T}}$. More generally, if we choose $x_1 = \alpha$, where α is any real parameter, then $x_2 = -4\alpha$. Therefore all solutions are given by

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \alpha \\ -4\alpha \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ -4 \end{pmatrix}.$$

Thus all solutions are multiples of $(1, -4)^{\mathrm{T}}$, and the solution set lies along the straight line through the origin defined by this vector. Geometrically, the two equations represent two lines in the plane that coincide.

Finally we introduce the notion of linear dependence and linearly independence of column vectors. Let $\mathbf{v}_1$, $\mathbf{v}_2$ be a set of 2 column vectors of the same size (both 2-vectors). We say the set is **linearly dependent** if there are two constants c_1, c_2, not both zero, for which

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{0}.$$

If this is true, then one of the vectors could be written as a multiple of the other; hence the word "dependence". If, on the other hand, this combination of $\mathbf{v}_1$and $\mathbf{v}_2$ forces both constants to be zero, then the set of vectors is not linearly dependent and they are called **linearly independent**. Hence, two vectors are linearly independent if one is not a multiple of the other. A set of three or more 2-vectors in the plane must be linearly dependent.

In the sequel we shall also need the notion of linear independence for **vector functions**. From multivariable calculus that a vector function in two dimensions has the form of a 2-vector whose entries are functions of time t, i.e.,

$$\mathbf{r}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix},$$

where t belongs to some interval I of time. The vector function $\mathbf{r}(t)$ is usually called the position vector, and its head traces out a curve in the plane given by the parametric equations $x = x(t)$, $y = y(t)$, $t \in I$. A set of two such vector functions $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$ are **linearly dependent** on I if one is a multiple of the other; stated differently, they are linearly independent if there are two constants c_1 and c_2, not both zero, for which

$$c_1\mathbf{r}_1(t) + c_2\mathbf{r}_2(t) = 0, \quad t \in I.$$

The set of two vector functions is **linearly independent** on I if it is not a linearly dependent set; so, $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$ are linearly independent on I if the last expression forces both constants c_1 and c_2 to be zero. Unlike scalar vectors (2-vectors with constant entries), a set of three or more vector functions in the plane need not be a linearly dependent set.

Example 5.9 The two vector functions

$$\mathbf{r}_1(t) = \begin{pmatrix} e^{2t} \\ 7 \end{pmatrix}, \quad \mathbf{r}_2(t) = \begin{pmatrix} 5e^{2t} \\ \sin t \end{pmatrix}$$

are linearly independent because neither is a multiple of the other.

Exercise 5.10 The three vector functions

$$\mathbf{r}_1(t) = \begin{pmatrix} e^{2t} \\ 7 \end{pmatrix}, \quad \mathbf{r}_2(t) = \begin{pmatrix} 5e^{2t} \\ \sin t \end{pmatrix} \quad \mathbf{r}_3(t) = \begin{pmatrix} 1 \\ \cos t \end{pmatrix},$$

are a linearly independent set because neither can be written as a combination of the others. That is, if we take a combination of them and set it equal to zero, i.e., $c_1\mathbf{r}_1(t) + c_2\mathbf{r}_1(t) + c_3\mathbf{r}_1(t) = 0$, for all t in $\mathbb{R}$, then we are forced to take $c_1 = c_2 = c_3 = 0$. (see Exercise 8).

Exercises

1. Let
$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 0 \\ 3 & 7 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 2 \\ -5 \end{pmatrix}.$$
Find $A+B$, $B-4A$, AB, BA, A^2, $B\mathbf{b}$, A^{-1}, $\det B$, B^3, AI, A^Tb, adj A, and $A - \lambda I$ (where λ a parameter).

2. Referring to Exercise 1, solve the system $A\mathbf{x} = \mathbf{b}$ using A^{-1}.

3. Find all values of the parameter λ that satisfy the equation $\det(A - \lambda I) = 0$, where A is given in Exercise 1.

4. Let
$$A = \begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix}.$$
Compute $\det A$. Does A^{-1} exist? Find all solutions to $A\mathbf{x} = \mathbf{0}$ and plot the solution set in the plane.

5. Determine all values m for which the system
$$2x + 3y = m, \quad -6x - 9y = 5,$$
has no solution, a unique solution, or infinitely many solutions.

6. Use the definition of linear dependence to show that the two vectors $(2,-3)^T$ and $(-4,6)^T$ are linearly dependent.

7. Let
$$A = \begin{pmatrix} 0 & 2 & -1 \\ 1 & 6 & -2 \\ 2 & 0 & 3 \end{pmatrix}, x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

(a) Find $A\mathbf{x}$.

(b) Find $\det A$. Is A invertible? How many solutions does the system $A\mathbf{x} = \mathbf{0}$ have?

8. Verify the claim in Example 5.10 (take two different values of t).

9. Plot each of the following vector functions in the xy plane, where $-\infty < t < +\infty$:

$$\mathbf{r}_1(t) = \begin{pmatrix} 3\cos t \\ 2\sin t \end{pmatrix}, \quad \mathbf{r}_2(t) = \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{2t}, \quad \mathbf{r}_3(t) = \begin{pmatrix} t \\ t+1 \end{pmatrix} e^{-t}.$$

Show that these vector functions form a linearly independent set.

10. Construct simple homogeneous systems $A\mathbf{x} = \mathbf{0}$ of three equations in three unknowns that has: (a) a unique solution, (b) an infinitude of solutions lying on a line in $\mathbb{R}^3$, (c) an infinitude of solutions lying on a plane in $\mathbb{R}^3$. Is there a case when there is no solution?