Series Tests Examples

Math 107

April 14, 2018

- **n^{th}-Term Test:** Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{1+3^n}$. Then

$$\lim_{n\to\infty} \frac{3^n}{1+3^n} \overset{\text{L'H}}{=} \lim_{n\to\infty} \frac{\ln(3)3^n}{\ln(3)3^n} = \lim_{n\to\infty} 1 = 1 \neq 0.$$

Therefore the series diverges by the n^{th}-Term Test since the limit of the summands does not equal zero. Note that we are allowed to use L'Hospital's Rule here since

$$\lim_{n\to\infty} 3^n = \lim_{n\to\infty} (1+3^n) = \infty.$$

- **n^{th}-Term Test:** Consider the series $\displaystyle\sum_{n=1}^{\infty} \cos(n)$. Note $\displaystyle\lim_{n\to\infty} \cos(n) \neq 0$ since the limit does not exist. Thus the series diverges by the n^{th}-Term Test.

- **Integral Test:** Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1+n^2}$. Observe that $n^2 > 0$ and so $1+n^2 > 0$ and so $\dfrac{1}{1+n^2} > 0$. Thus our summands are positive.

Now observe that

$$\frac{1}{1+(n+1)^2} = \frac{1}{n^2+2n+2} < \frac{1}{1+n^2}$$

since $n^2+2n+2 > 1+n^2$. Therefore our summands are decreasing as well as being positive. Thus we may use the Integral Test by observing the improper integral $\displaystyle\int_1^{\infty} \frac{1}{1+x^2}\,\mathrm{d}x$. Recall

$$\int_1^{\infty} \frac{1}{1+x^2}\,\mathrm{d}x = \lim_{b\to\infty} \arctan(x)]_1^b = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}.$$

Therefore the improper integral $\displaystyle\int_1^{\infty} \frac{1}{1+x^2}\,\mathrm{d}x$ converges. Thus by the Integral Test, the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1+n^2}$ converges as well.

- **Integral Test:** Consider the series $\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$. Observe that $n \geq 2$ and so $\ln(n) > 0$ and so $\frac{1}{n\ln(n)} > 0$. Thus our summands are positive.

  Now observe that

$$\frac{1}{(n+1)\ln(n+1)} = \frac{1}{n\ln(n+1) + \ln(n+1)} < \frac{1}{n\ln(n)}$$

  since $n\ln(n+1) + \ln(n+1) > n\ln(n)$. Thus our summands are decreasing as well as positive. Thus we may use the Integral Test by observing the improper integral $\int_2^{\infty} \frac{1}{x\ln(x)}\,\mathrm{d}x$. Recall

$$\int_2^{\infty} \frac{1}{x\ln(x)}\,\mathrm{d}x = \lim_{b\to\infty} \ln(\ln(x))]_2^b = \infty.$$

  Therefore the improper integral $\int_2^{\infty} \frac{1}{x\ln(x)}\,\mathrm{d}x$ diverges. Thus by the Integral Test, we know that the series $\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$ diverges as well.

- ***p*-Test:** The series $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}}}$ diverges by the p-Test since $\frac{1}{2} \leq 1$.

- **Geometric Series:** Consider $\sum_{n=0}^{\infty}(-3)^{-n}$. Observe that $(-3)^{-n} = \left(-\frac{1}{3}\right)^n$ and the series $\sum_{n=1}^{\infty}\left(-\frac{1}{3}\right)^n$ converges since $\left|-\frac{1}{3}\right| < 1$. Specifically, we know that the series converges to $\frac{1}{1-(-\frac{1}{3})} = \frac{3}{4}$.

- **Geometric Series Test:** Consider $\sum_{n=0}^{\infty}\left(-\frac{\pi}{e}\right)^n$. This series diverges since $\left|-\frac{\pi}{e}\right| \geq 1$.

- **Direct Comparison Test:** Consider the series $\sum_{n=1}^{\infty} \frac{5+2\cos(n)}{n}$. Note that we know $-1 \leq \cos(n) \leq 1$ and therefore $-2 \leq \cos(n) \leq 2$ and so $3 \leq 5 + 2\cos(n) \leq 7$. Therefore $\frac{3}{n} \leq \frac{5+2\cos(n)}{n} \leq \frac{7}{n}$. Note that $\sum_{n=1}^{\infty} \frac{3}{n}$ diverges by the p-Test. Therefore $\sum_{n=1}^{\infty} \frac{5+2\cos(n)}{n}$ diverges by the Direct Comparison Test, since it is greater than or equal to a positive series that diverges.

- **Limit Comparison Test:** Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{2n^2+60n+1}{3n^5-800n^4+n^3-7n-6}$. Let us compare this with the simpler series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$. We see that

$$\lim_{n\to\infty} \left(\frac{2n^2+60n+1}{3n^5-800n^4+n^3-7n-6}\right) / \left(\frac{1}{n^3}\right) = \lim_{n\to\infty} \left(\frac{2n^5+60n^4+n^3}{3n^5-800n^4+n^3-7n-6}\right)$$
$$= \frac{2}{3} > 0.$$

Therefore the two series do the same thing. Since we already know that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges (by the p-Test or Integral Test), we may then conclude that the series $\displaystyle\sum_{n=1}^{\infty} \frac{2n^2+60n+1}{3n^5-800n^4+n^3-7n-6}$ converges as well.

- **Ratio Test:** Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$. Then

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{((n+1)!)^2}{(2(n+1))!} \cdot \frac{(2n)!}{(n!)^2}\right| = \lim_{n\to\infty} \left|\frac{(2n)!(n+1)!(n+1)!}{(2n+2)!(n)!(n)!}\right|$$
$$= \lim_{n\to\infty} \left|\frac{(2n)!}{(2n+2)!} \cdot \frac{(n+1)!}{n!} \cdot \frac{(n+1)!}{n!}\right| = \lim_{n\to\infty} \left|\frac{(n+1)(n+1)}{(2n+2)(2n+1)}\right| = \frac{1}{4} < 1.$$

Thus the series converges by the Ratio Test.

- **Ratio Test:** Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{(n+1)^2 2^n}$. Using the Ratio Test, we can see that

$$\lim_{n\to\infty} \left|\frac{(n+1)!}{(n+2)^2 2^{n+1}} \cdot \frac{(n+1)^2 2^n}{n!}\right| = \lim_{n\to\infty} \left|\frac{(n+1)(n+1)^2}{2(n+2)^2}\right|$$

diverges. Therefore the series diverges.

- **Alternating Series Test:** Consider the series $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2+1}$. We know that $n > 0$ and so $n^2+1 > 0$ and so $\dfrac{n}{n^2+1} > 0$. Furthermore, we can see that $\frac{n+1}{(n+1)^2+1} < \frac{n}{n^2+1}$ since $(n^2+1)(n+1) < ((n+1)^2+1)n$ since $n^3+n^2+n+1 < n^3+2n^2+2n$. Thus the absolute value of our summands is decreasing.

Now we can use L'Hospital's Rule to show that $\displaystyle\lim_{n\to\infty} \frac{n}{n^2+1} = \lim_{n\to\infty} \frac{1}{2n} = 0$. Therefore by the Alternating Series Test, the series converges.

- **Alternating Series Test:** Consider the series $\sum_{n=2}^{\infty}(-1)^n\frac{\cos(\frac{1}{n})}{n}$. First note that $\cos(x)$ is positive for when $x \leq \frac{\pi}{2}$ and $\frac{1}{n} < 1 < \frac{\pi}{2}$ for all $n \geq 2$. Therefore $\cos(\frac{1}{n})$ is positive. Furthermore, $\frac{\cos(\frac{1}{n})}{n}$ is then positive.

  To show that $\frac{\cos(\frac{1}{n})}{n}$ is decreasing, let us observe that the derivative is

  $$\frac{1}{n^3}\sin\left(\frac{1}{n}\right) - \frac{1}{n^2}\cos\left(\frac{1}{n}\right) = \frac{\sin(\frac{1}{n}) - n\cos(\frac{1}{n})}{n^3}.$$

  Since $\sin(x) < \cos(x)$ for $0 < x < \frac{\pi}{4}$ and since $\frac{1}{n} < \frac{\pi}{4}$ for all $n \geq 2$, we know that $\sin(\frac{1}{n}) < \cos(\frac{1}{n})$ and so $\sin(\frac{1}{n}) < n\cos(\frac{1}{n})$ and so $\sin(\frac{1}{n}) - n\cos(\frac{1}{n}) < 0$. Therefore the derivative is negative and so our summands are decreasing.

  Now observe that $\lim_{n\to\infty}\frac{\cos(\frac{1}{n})}{n} = \lim_{n\to\infty}\cos\left(\frac{1}{n}\right) \times \lim_{n\to\infty}\frac{1}{n} = 1 \times 0 = 0$. Thus by the Alternating Series Test, $\sum_{n=2}^{\infty}(-1)^n\frac{\cos(\frac{1}{n})}{n}$ converges.

- **Absolute Convergence:** Consider the series $\sum_{n=1}^{\infty}\frac{\cos(n)}{n^2}$. We know that $\left|\frac{\cos(n)}{n^2}\right| \leq \frac{1}{n^2}$. Therefore $\sum_{n=1}^{\infty}\left|\frac{\cos(n)}{n^2}\right|$ converges by the Direct Comparison Test. Therefore $\sum_{n=1}^{\infty}\frac{\cos(n)}{n^2}$ converges absolutely.