Summer 2017 CURL Research

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1. Paths, Cycles, and Wheels

We continue some leftover material from the matching complex group's spring semester work. First we reproduce the results on homology for paths with a different method. Then we do some work with wheel graphs.

Definition 1.1. Let $G = (V, E)$ be a graph. Then if $S \subset V$, $G[V-S]$ is the subgraph of G induced by the vertex set $V - S$

Definition 1.2. The *matching complex of a graph* G is the simplicial complex whose elements are subgraphs of G in which no edge shares a vertex. We denote the matching complex as $M(G)$

Recall the following long exact sequence and theorem:

Theorem 1.3. *For any graph G and a fixed edge 12 in the edge set of G, the following long sequence is exact,*

$$\cdots \xrightarrow{\delta} H_{t-1}(\bigoplus_{a,i} M(G[V-\{1,2,i\}])) \xrightarrow{\phi} H_t(M(G)) \xrightarrow{\psi} H_{t-2}(\bigoplus_{i,j} M(G[V-\{1,2,i,j\}])) \xrightarrow{\delta} H_{t-2}(\bigoplus_{a,i} M(G[V-\{1,2,i\}])) \xrightarrow{\phi} \cdots$$

The sum $\bigoplus_{a,i} M(G[V-\{1,2,i\}])$ is over all edges ai where $a \in \{1,2\}$ and $i \in \{3,...,n\}$. The sum $\bigoplus_{i,j} M(G[V-\{1,2,i,j\}])$ is over all pairs of edges $1i, 2j$ such that $i \neq j \in \{3,...n\}$.

We also had the following theorem as a direct result [1, Theorem 11.42]:

Theorem 1.4. *Let $v_n = \lfloor \frac{n-2}{3} \rfloor$, then the homology for the matching complex for the path graph on n vertices P_n when $n = 0, 2$ mod 3 is*

$$H_i(MP_n) = \begin{cases} 0 & \text{if } i \neq v_n \\ \mathbb{Z} & \text{if } i = v_n \end{cases}$$

When $n = 1$ mod 3,

$$H_i(MP_n) = 0$$

for all i

We provide a proof of 1.4 by using 1.3 proven in the other matching complex report. The original proof used decision trees and gave us the topological realizations of the complex but we would just like to demonstrate and check the exact sequence works with paths.

Proof. We use induction. The first cases for $n = 0, 1, 2, 3$ are trivial or easily checked. Then for P_n label the vertices such vertex 1 has degree 1 and vertex i is adjacent to vertex $i-1$ and $i+1$ for $1 < i < n$. Then vertex 1 has no neighbors other then vertex 2 and vertex 2 only has one neighbor, vertex 3, other then vertex 1 so the sums from 1.3, $\bigoplus_{a,i} M(G[V-\{1,2,i\}]) = MP_{n-3}$ and $\bigoplus_{i,j} H_{t-2}(M(G[V-\{1,2,i,j\}]))$ is trivial and so will have homology 0 everywhere in our long exact sequence. So then we get that $H_{i-1}(MP_{n-3}) = H_i(MP_n)$ and in particular it is nonzero when $H_{v_{n-3}}(MP_{n-3}) = H_{v_n-1}(MP_{n-3}) = H_{v_n}(MP_n)$. $\square$

Now we prove a statement on vertex decomposability of Wheel Graphs. We use the following definitions found in the previous matching complex report. Then we define a new graph which will be the wheel graph and cycle graph in special cases.

Definition 1.5. Given a simplicial complex Δ and an element $\sigma \in \Delta$, we define the following,

(1) $link_\Delta(\sigma) = \{\tau \in \Delta | \tau \cap \sigma = \emptyset \text{ and } \tau \cup \sigma \in \Delta\}$
(2) $del_\Delta(\sigma) = \{\tau \in \Delta | \tau \cap \sigma = \emptyset\}$
(3) $fdel_\Delta(\sigma) = \{\tau \in \Delta | \sigma \not\subset \tau\}$

Definition 1.6. We define the class of *vertex-decomposable (VD)* lifted complexes recursively as follows[1, Definition 3.27]:
(1) Every simplex (including $\emptyset$ and $\{\emptyset\}$ is VD).
(2) If Δ contains a vertex v such that $\Delta(v, \emptyset)$ and $\Delta(\emptyset, v)$ are VD of the same dimension, then Δ is also VD.
(3) If Δ is a cone over a VD complex Δ', then Δ is also VD.
(4) If $\Delta = \Sigma * \{\sigma\}$ and Σ is VD, then Δ is also VD.

This is defined for general complexes. Moreover, we say a complex is *VD(d)* if the d-skeleton is VD. There is an important lemma about this class of complexes.

Lemma 1.7. *Let Δ be a lifted complex and let v be a vertex. If $link_\delta(v)$ is VD(d-1) and $del_\delta(v)$ is VD(d), then δ is VD(d).*

We will use a previous result found in Johnson [1, Theorem 11.42, 11.43] which the following is a part of

Theorem 1.8. *Let $v_n = \lfloor \frac{n-2}{3} \rfloor$, both the cycle graph C_n and the path graph P_n are $VD(v_n)$*

Then we define a new graph which will connect wheel graphs and cycles.

Definition 1.9. Let $n > 4$ and $0 \leq k < n$. Then construct the following graph, take a cycle of $n-1$ vertices and label the vertices such that $(i-1)i$ and $i(i+1)$ are edges modulo $n-1$. Then have a additional vertex labeled n and an edge $i(n+1)$ for all $i \leq k$. Call this graph W_n^k

Notice that W_k^0 is the cycle graph on $n-1$ and an additional vertex. The matching complex is the same as the for the matching complex of C_{n-1}. Also W_k^{n-1} is the wheel graph on n vertices.

Theorem 1.10. *Let v_n be the same as the one for cycle graphs and paths in Theorem 1.4. W_n^K is $VD(v_{n-1})$.*

Proof. We use induction. For the base case as explained above $MW_n^0 = MC_{n-1}$ so it is $VD(v_{n-1})$. Then for $k > 0$, choose edge kn in graph W_n^k. Then $link_{WD_n^k}(kn) = MP_n - 2$ which is $VD(v_{n-2})$ so it is $VD(v_{n-1} - 1)$ as well, and $del_{WD_n^k}(kn) = MW_n^{k-1}$ which is $VD(v_{n-1})$ by induction. So by 1.7, W_n^k and in particular the wheel graph is $VD(v_{n-1})$. □

We are able to derive the desired results for the homology of the wheel graph. We will use decision trees to find the number of evasive sets and that will tell us information about our complexes. We will use the W_n^k graph defined in definition 1.9.

Definition 1.11. [1, Definition 5.1] The class of *element-decision trees*, each associated to a finite family of finite sets, is defined recursively as follows:

(1) $T =$ **Win** is an element-decision tree on $\emptyset$ and on any 0-simplex $\{\emptyset, \{v\}\}$.
(2) $T =$ **Lose** is an element-decision tree on $\{\emptyset\}$ and on any singleton set $\{\{v\}\}$.
(3) If Δ is a family of sets, if x is an element, if T_0 is an element-decision tree on $del_\Delta(x)$, and if T_1 is an element-decision tree on $link_\Delta(x)$, then the triple (x, T_0, T_1) is an element-decision tree on Δ.

Then we can define a non-evasive (and evasive) set.

Definition 1.12. [1, p.70] Given a simplicial complex Δ and a decision tree T of Δ, a set $\tau \in \Delta$ is *nonevasive* if any of the following hold

(1) $T =$ **Win**
(2) $T = (x, T_0, T_1)$ for some $x \notin \tau$ and τ is nonevasive with respect to T_0.
(3) $T = (x, T_0, T_1)$ for some $x \in \tau$ and $\tau - x$ is nonevasive with respect to T_1.

A simplex is **evasive** if it is not non-evasive. We can replace the nonevasive with evasive in condition (2) and (3) and it will still hold.

Then the following theorems from Forman's work on Morse Theory will be useful. These form of the theorems are taken from Jonnson's work [1] but is originally from Forman's paper[2]. We further modified the language in places since we are just working with simplicial complexes.

Definition 1.13. Given a simplicial complex Δ we say M is a *matching on* Δ if the elements of M are pairs of simplicies of Δ such that each simplex is in at most one pair and each pair is of form (σ, τ) where $\sigma, \tau \in \Delta$ and $\tau = \sigma \cup x$, $x \notin \sigma$.

We call simplices that are unmatched in a matching of a simplicial complex *critical* or *unmatched*.

Then given a matching M of Δ, we can draw a graph whose vertices are simplices and there is an edge between σ and τ if $\tau = \sigma \cup x$ for some $x \notin \sigma$. Then for all edge pairs in the matching M, orient the edge as (σ, τ). For all remaining edges orient it as (τ, σ). We say a matching is **acyclic** if there is no directed cycle $e_1, e_2, ..., e_n$ such that for all $i \in [n]$, e_i only contains simplices of dimension k and $k+1$ for some k.

Theorem 1.14. [1, Theorem 5.2] *Let Δ be a simplicial complex and T a decision tree of Δ. Then there is an acyclic matching on Δ such that the critical sets are precisely the evasive sets in Δ with respect to T. Furthermore Δ is homotopy equivalent to a CW complex with exactly one cell of dimension p for each evasive set in Δ of dimension p and one additional 0-cell.*

The following theorem is particularly useful in our case and will give us a lot of information about matching complexes of wheel graphs.

Theorem 1.15. [1, Theorem 4.8] *Let Δ be a simplicial complex. If all critical faces of Δ are of the same dimension d, then Δ is homotopy equivalent to a wedge of spheres of dimension d.*

Theorem 1.16. [1, Corollary 4.6] *If a simplicial complex Δ does not contain any critical faces, then Δ is collapsible and hence contractable to a point.*

By 1.14 the number of spheres wedged together by 1.15 is the same as the number of critical points. Furthermore the homology of spheres and spheres wedged together are known.

Theorem 1.17. *The homology of an n-dimensional sphere S^n is*

$$H_k(S^n) = \begin{cases} \mathbb{Z} & \text{k=0,n} \\ \{0\} & \text{else} \end{cases}$$

The wedge of parts under taking homology will be a direct sum of the homology of the parts. In particular the homology of a wedge of spheres is the sum of the homology of each individual sphere

The following is part of the proof for the homology of matching complexes for paths P_n and cycles C_n[1, Theorem 11.42,11.43].

Theorem 1.18. *There exists an element decision tree for MP_n, such that there is one evasive simplex of dimension v_n when $n \equiv 0, 1$ mod 3 and no evasive faces when $n \equiv 2$ mod 3*
Furthermore there exists an element decision tree for MC_n, such that there is one evasive simplex of dimension v_n for $n \equiv 1, 2$ mod 3 and two evasive faces of dimension v_n when $n \equiv 0$ mod 3.

Now we will prove the homology results for wheel graphs. For notation we mark the wedge of k copies of some space X as $X^{\wedge k}$.

Proposition 1.19. For $n > 3$, $k < n$, and $v_n = \lfloor \frac{n-2}{3} \rfloor$,

$$MW_n^k = \begin{cases} S^{v_n} \wedge S^{v_n} & \text{when } n \equiv 1 \bmod 3 \\ (S^{v_n})^{\wedge k+1} & \text{when } n \equiv 0 \bmod 3 \\ (S^{v_n})^{\wedge k-1} & \text{when } n \equiv 2 \bmod 3 \end{cases}$$

We let $(S^{v_n})^{\wedge 0}$ be a point.

Proof. Label the vertices of W_n^k as in Definition 1.9. We use induction on k up to the value of $n-1$.

First we consider when $n \equiv 1 \bmod 3$. We get an element decision tree of MW_n^1 as $(1n, T_0, T_1)$, with T_0 an element decision tree for $del_\Delta(1n) = MW_n^0 = MC_{n-1}$ and T_1 an element decision tree for $link_\Delta(1n) = MPa_{n-2}$. From Theorem 1.18 we know there is a decision tree T_1 with no evasive sets since $n-2 \equiv 2 \bmod 3$ and T_0 with two evasive sets of dimension $v_n = v_{n-1}$ since $n-2 \equiv 0 \bmod 3$. So MW_n^1 will have two evasive faces dimension v_n

Then for MW_n^k, we get the element decision tree (kn, T_0, T_1) where T_0 is an element decision tree for $del_\Delta(1n) = MW_n^{k-1}$ and T_1 is an element decision tree for $link_\Delta(1n) = MPa_{n-2}$. There exists a T_1 that also has no evasive sets again and by induction T_0 has two evasive sets of dimension v_n. Then by Theorem 1.14 there are exactly two critical simplicies of dimension v_n and Theorem 1.15 gives us the Proposition.

For $n \equiv 0 \bmod 3$, we get an element decision tree of MW_n^1 as $(1n, T_0, T_1)$, with T_0 an element decision tree for $del_\Delta(1n) = MW_n^0 = MC_{n-1}$ and T_1 an element decision tree for $link_\Delta(1n) = MPa_{n-2}$ again. This time there exists T_0 that has one evasive set of dimension $v_{n-1} = v_n$ and T_1 that has one evasive set of dimension $v_{n-2} = v_n - 1$. Then if τ is an evasive set of T_1 then $\tau + 1n$ is an evasive set of MW_n^1. So MW_n^1 has 2 evasive sets of dimension v_n.

Then for MW_n^k, we again get the element decision tree (kn, T_0, T_1) where T_0 is an element decision tree for $del_\Delta(1n) = MW_n^{k-1}$ and T_1 is an element decision tree for $link_\Delta(1n) = MPa_{n-2}$. MW_n^{k-1} will have a tree with k evasive sets dimension v_n and MPa_{n-2} will have a tree with an evasive set of dimension $v_n - 1$ again. Then by the same argument for the base case MW_n^k will have $k+1$ evasive sets dimension v_n. 1.14 and 1.15 again will give us the proposition.

Finally consider $n \equiv 2 \bmod 3$. We will use a different decision tree for MW_n^1 this time. We look at edge 12 instead of $1n$. Then we get an element tree of MW_n^1, $(12, T_0, T_1)$, where T_0 is an element decision tree for $del_\Delta(12) = MP_n$ and T_1 is an element decision tree for $link_\Delta(12) = MPa_{n-3}$. Then since $n \equiv n-3 \equiv 2 \bmod 3$, by 1.18 we know there is a tree with no evasive sets. So MW_n^1 collapses to a single point.

The rest of the proof follows as before. We get the element decision tree (kn, T_0, T_1) where T_0 is an element decision tree for $del_\Delta(1n) = MW_n^{k-1}$ and T_1 is an element decision tree for $link_\Delta(1n) = MPa_{n-2}$. The tree for MW_n^{k-1} this time has $k-2$ evasive sets of dimension v_n and the exists a tree for MPa_{n-2} has one with dimension $v_n - 1 = v_{n-2}$. Put them together to get $k-1$ evasive sets dimension v_n. □

Then we get homology results as a direct corollary using 1.17.

Corollary 1.20. *For $n > 3$ and $k < n$ $v_n = \lfloor \frac{n-2}{3} \rfloor$, for $n \equiv 1$ mod 3*

$$H_i(MW_n^k) = \begin{cases} \mathbb{Z}^2 & i = v_n \\ 0 & else \end{cases}$$

For $n \equiv 0$ mod 3,

$$H_i(MW_n^k) = \begin{cases} \mathbb{Z}^{k+1} = \mathbb{Z}^{3v_n+3} & i = v_n \\ 0 & else \end{cases}$$

For $n \equiv 2 \mod 3$*,*

$$H_i(MW_n^k) = \begin{cases} \mathbb{Z}^{k-1} = \mathbb{Z}^{3v_n} & i = v_n \\ 0 & else \end{cases}$$

Proof. When $n \equiv 0 \bmod 3$, $k+1 = 3v_n + 3$ and when $n \equiv 2 \bmod 3$, $k-1 = 3v_n$ □

Now we wish to write code that will compute if a set is evasive given a decision tree. Recall in Definition 1.11, we defined a decision tree inductively from the top down. So if we have a ordering of the 0-dimension simplicies, then it will define a decision tree. For example if we have a 3 0-dimensional simplicies, then if we have order 0,1,2 then we get a tree of the simplicial complex by decomposing as $(0, T_0, T_1)$. Then in the next step we would decompose the tree as $(1, T_0, T_1)$, and so on.

So first we will write code that will take a simplicial complex, order and a simplex as input and give us a boolean output on if it is evasive or not. We introduce another type of decision tree to make this a little easier. We replace the elements with sets and deletion with face deletion.

Definition 1.21. [1, Definition 5.3] The class of *set-decision trees*, each associated to a finite family of finite sets, is defined recursively as follows:

(1) $T =$ **Win** is an element-decision tree on $\emptyset$ and on any 0-simplex $\{\emptyset, \{v\}\}$.
(2) $T =$ **Lose** is an element-decision tree on $\{\emptyset\}$ and on any singleton set $\{\{v\}\}$.
(3) If Δ is a family of sets, if x is an element, if T_0 is an element-decision tree on $fdel_\Delta(x)$, and if T_1 is an element-decision tree on $link_\Delta(x)$, then the triple (x, T_0, T_1) is an element-decision tree on Δ.

We define evasive sets on the tree in the same way as Definition 1.12. Then we have a similar theorem to Theorem 1.14.

Theorem 1.22. [1, Theorem 5.4] *Let* Δ *be a finite simplicial complex and* T *be a set-decision tree on* Δ*. Then there is an acyclic matching on* Δ *such that the critical sets are precisely the evasive sets in* Δ *with respect to T. Conversely, given an acyclic matching M on* Δ*, there is a set-decision tree on* Δ *such that the evasive sets are precisely the critical sets with respect to M.*

So now the theorem contains a converse and we can go back and forth between set-decision trees and acyclic matchings. In particular, we know that a simplex is topologically realized as a ball of the same dimension. This is homotopy equivalent to a point and so it is contractable and has a acyclic matching. This will allow us when trying to determine if a set is non-evasive to ask if the resulting complex from taking the link or deletion is a simplex instead of empty or $\{\emptyset, \{v\}\}$.

The code is provided in Section 13.

2. Decision Tree for Complete and r-partite matching complexes

Definition 2.1. Given a graph G = (V,E) and a subset S of the edges E let $G[E-S]$ be the subgraph of G with the edges in S deleted. $G[E-S] = G(V, E-S)$

We will also use a fact from discrete morse theory that will give us a bound for the dimension of the homology.

Theorem 2.2. *Given any acyclic matching of a simplicial complex* Δ*, with k critical faces of dimension i, then* $\dim(H_i(\Delta)) \leq k$

We will now look at decision trees of the complete graph. Let us label the verteces of the complete graph on n vertices K_n from 1 to n. Then we will first consider edge 12. The link will remove all edges containing vertex 1 or 2, so $link_{MK_n}(12) = MK_{n-2}$. The deletion will give us $M(K_n(E-\{12\}))$, the matching complex of the complete graph without edge 12. Then we will further decompose the deletion. Next we consider edge 13 in $M(K_n(E-\{12\}))$. The link will remove all edges connected to vertex 1 or 3. Notice that the edge 12 we previously deleted from our graph would be deleted from the graph as well if it were still there. This means that $link_{M(K_n(E-\{12\}))}(13) = link_{M(K_n)}(13) = MK_{n-2}$. Then we also take the deletion again to get $M(K_n(E-\{12,13\}))$. We can continue doing this on edges $14, 15, ..., 1n$ and this will give a decision tree with $n-1$ copies of M_{n-2} and a copy of M_{n-1}. We can then replace MK_{n-1} and MK_{n-2} with their respective trees. This gives us the following by adding in trees that will minimize the number of evasive sets of a given dimension:

Theorem 2.3. *Let $ev(k,n)$ be the minimum number of evasive sets dimension k possible from a decision tree of MK_n. Then*

$$\dim(H_i(MK_n)) \leq ev(i,n) \leq (n-1)ev(i-1,n-2) + ev(i,n-1)$$

If there is a single set decision tree such that it has $ev(i,n)$ evasive sets of dimension i for all i and $\dim(H_i(MK_n)) = ev(i,n)$, we call the complex **semi-collapsible** (over some field or $\mathbb{Z}$). We also call this decision tree the optimal decision tree. There are propositions however that will tell us that complex is not semi-collapsible in most cases.

Proposition 2.4 ([1] Proposition 5.9,5.10)**.** If a complex is semi-collapsible over $\mathbb{Q}$ then it is semi-collapsible over $\mathbb{Z}$. Furthermore if the complex is semi-collapsible over $\mathbb{Q}$ then the $\mathbb{Z}$ homology is free.

We know that torsion does appear in the $\mathbb{Z}$-homology of certain matching complexes so it will not be semi-collapsible.

We can do the same process of constructing a decision tree for matching complexes of bipartite and r-partite graphs in the same manner. Again this will give us a bound for homology in terms of minimal evasive sets.

3. 3-partite Graphs

We have the following partial result. For 3-partite graphs we proved that it vanishes for $v_{r,s,t}-1$, $v_{r,s,t} = min\{\frac{r+s+t+1/3}{3}, s\}$. We will assume for now and hopefully prove later the case when $2s < r+t$. This is when s is the minimum and not $\frac{r+s+t+1/3}{3}$.

The following theorem gives the generators of matching complexes of complete graphs.

Theorem 3.1. [6, Lemma 2.5] *Suppose $n = 0, 1$ mod 3. Then $H_{\nu_n}(M_n)$ is generated by elements of the form*

$$(\sigma(1)\sigma(2) - \sigma(1)\sigma(3)) \wedge (\sigma(4)\sigma(5) - \sigma(4)\sigma(6)) \wedge ... \wedge (\sigma(N-2)\sigma(N-1) - \sigma(N-2)\sigma(N))$$

where $\sigma \in S_n$ and $N = 3\lfloor n/3 \rfloor$. Here wedge denotes concatenating disjoint elements together to make a larger matching.

Theorem 3.2. *Let $r \geq s \geq t$ and $2s \geq r+t$(Assume the case otherwise for now...). Then homology of $M_{r,s,t}$ does not vanish at $v_{r,s,t}$.*

Proof. We use induction. We know that the case for $s=0$ and $t=0$ holds. Then for $r+s+t \equiv 0, 1$ mod 3, we embed $M_{r,s,t}$ into M_{r+s+t}. This can be done since $K_{r,s,t}$ is a subgraph of K_{r+s+t}. This injection will induce a function $H_t(M_{r,s,t}) \to H_t(M_{r+s+t})$. So if we can find a element of $H_t(M_{r,s,t})$ whose image is the one mentioned in Theorem 3.1 then we will have found a non-trivial element of the homology group of $M_{r,s,t}$. So we will construct the generator as follows. Let C_1, C_2, C_3 be the components of the 3-partite graph $K_{r,s,t}$. The vertices are $C_1 = \{x_1, x_2, ..., x_r\}$,$C_2 = \{y_1, y_2, ..., y_s\}$, and $C_3 = \{z_1, z_2, ..., z_t\}$. Then for as many vertices $y_i \in C_2$ we match it with two vertices from either $x_j, z_k \in C_1, C_3, x_j \neq x_k \in C_1$, or $z_j \neq z_k \in C_1, C_3$ so we have triples $y_ix_j - y_iz_k$,$y_ix_j - y_ix_k$, or $y_iz_j - y_iz_k$.

In the case there is a single vertex v in C_1 or C_3 left unpaired then make the triple $vy_i - vy_j$ where $y_i\ y_j$ are unmatched vertices left in C_2. If this cannot be done then we have paired $N = 3\lfloor r+s+t/3 \rfloor$ of our vertices and we are done.

Then there will be $s - \lfloor \frac{r+t}{2} \rfloor$ unmatched vertices in C_2 remaining if $r+t$ is even and $s - \lfloor \frac{r+t}{2} \rfloor - 2$ unmatched vertices in C_2 remaining if $r+t$ is odd. Then take a vertex triple $xy_1 - zy_1$ already made with $y_1 \in C_2$ and $x, z \notin C_2$ already made. Then if we have $y_2, y_3, y_4 \in C_2$ unpaired vertices then we make the new pairings $xy_1 - xy_2$ and $zy_3 - zy_4$. Then we will be unable to do this when we have less then 3 unpaired vertices in C_2 so then we will have paired $N = 3\lfloor r+s+t/3 \rfloor$ of the vertices. Then take the sum and we have our generator.

For the case of $r + s + t \equiv 2 \bmod 3$ we look at the tail end of the exact sequence derived from Theorem 1.3

$$\cdots \xrightarrow{\phi} H_{v_{r,s,t}}(M(G)) \xrightarrow{\psi} H_{v_{r,s,t}-2}(\bigoplus_{i,j} M(G[V - \{1, 2, i, j\}])) \to 0$$

Then by induction $H_{v_{r,s,t}-2}(\bigoplus_{i,j} M(G[V - \{1, 2, i, j\}]))$ is non-zero so $H_{v_{r,s,t}}(M(G))$ must be non-zero. □

We now introduce more objects that will help us compute the homotopy equivalences for some complexes. We define these objects combinatorially but will often use theorems and properties known about their topological realization.

Definition 3.3. A *cone on a x* is a simplicial complex Δ such that each simplex σ either contains x or can be extended so that it contains x, i.e. $\sigma \cup x \in \Delta$.

Then we know an important fact on cones

Theorem 3.4. *Cones are homotopy equivalent to a point. It follows that their homology is trivial everywhere.*

4. Important notes from Reiner and Roberts

Definition 4.1. A *semi-group* is a set with an associative binary operation. A *sub-semigroup* is a subset of the semigroup closed under the binary operation of the semigroup.

Let Λ be a finitely generated additive sub-semigroup of $\mathbb{N}^d$ and let $\mathcal{M} \subseteq \mathbb{N}^d$ be a finitely-generated Λ-module. Consider the semigroup ring $k[\Lambda]$ (where k is any field). Then $k[\Lambda]$ may be identified with a subalgebra of $k[z_1, \dots, z_d]$ generated by monomials $m_1, \dots, m_n$. So we have the finitely generated module $M = k\mathcal{M}$ over $k[\Lambda]$ inside $k[\mathbf{z}]$ by taking the k-span of monomials of the form $\mathbf{z}^\mu$ with $\mu \in \mathcal{M}$. Surjecting $A = k[x_1 \dots, x_n]$ onto $k[\Lambda]$ via $x_i \mapsto m_i$, we endow $k[\Lambda]$ and M the structure of finitely generated A-modules. Given $\mu \in \mathcal{M}$ set $[n] := \{1, \dots, n\}$ then we have the simplicial complex:

$$K_\mu := \{F \subseteq [n] \mid \frac{\mathbf{z}^\mu}{\prod_{i \in F} m_i} \in M\} \text{ where } \mathbf{z}^\mu := z_1^{a_1} \dots z_d^{a_d}$$

Proposition 4.2. For any $\gamma \notin \mathcal{M}$, $\mathrm{Tor}_i^A(M, k)_\gamma = 0$ and we have

$$\mathrm{Tor}_i^A(M, k)_\mu \cong \tilde{H}_{i-1}(K_\mu; k)$$

Now Segre$(m, n, 0)$ is the semigroup ring for the submonoid of $\mathbb{N}^m \times \mathbb{N}^n$ generated by the set $\{(e_i, e_j) \mid 1 \leq i \leq m \text{ and } 1 \leq j \leq n\}$. Here e_k is the kth standard basis vector. Veronese(n, 2, 0) is the semigroup ring for the submonoid of $\mathbb{N}^n$ generated by $\{(e_i + e_j)\}_{1 \leq i \leq j \leq n}$. In general, for any multidegree γ in Veronese$(n, 2, r)$ the complex K_γ can be identified with the bounded degree graph-complex Δ_γ whose vertices correspond to the possible loops and edges in a complete graph on n vertices, and whose faces are the subgraphs (with loops allowed) in which the degree of vertex i is bounded by γ_i. Here a loop on a vertex is counted as adding 2 to the degree of the vertex. (So setting $\gamma = (1, \dots, 1)$ we have $\Delta_\gamma = \Delta_n$ the matching complex on n vertices's).

Definition 4.3. The *Chessboard complex with multiplicities* $\Delta_{\gamma,\delta}$ is defined to be the simplicial complex with a vertex set of the squares on an $m \times n$ chessboard and whose simplices are the sets of squares having no more than γ_i squares from row i and no more than δ_j squares on row j.

When the weighting $\gamma = (1, \dots, 1)$ the complex K_γ (the K_μ complex above with $\mu = \gamma$) is the matching complex Δ_n for a complete graph on n vertices.

Proposition 4.4.

$$\mathrm{Tor}_i^{A_{m,n}}(Segre(m, n, r), k)_{(\gamma,\delta)} \cong \tilde{H}_{i-1}(\Delta_{\gamma,\delta}; k)$$

$$\mathrm{Tor}_i^{A_n}(Veronese(n, 2, r), k)_\gamma \cong \tilde{H}_{i-1}(\Delta_\gamma; k)$$

And for $|\gamma| > 2i$ all torsion groups vanish. (Here $A_{m,n} = k[x_{ij}]$ with $1 \le i \le m$ and $1 \le j \le n$ and $A_n = k[x_1, \dots, x_n]$, so by surjecting $A_{m,n}$ and A_n onto the semigroup ring we give the semigroup ring a finitely generated $A_{m,n}$-module and A_n-module structure)

So if we only consider the square free multidegree's $(\gamma, \delta) = ((1, \dots, 1), (1, \dots, 1))$ and $\gamma = (1, \dots, 1)$ we have that $\Delta_{\gamma,\delta} = \Delta_{m,n}$ so we can relate the torsion groups of the Segre chain complex to the reduced homology groups of the standard chessboard complex and similarly the torsion groups of certain Veronese chain complexes to reduced homology groups of standard matching complexes. However by varying these multidegree's we get a correspondence with the chessboard complex with multiplicities and multi-matching complex's, respectively.

5. Representation theory

Definition 5.1. The polynomial ring $S = k[x_1, x_2, \dots, x_n]$ is multigraded by A if it has been given a degree map $deg : \mathbb{Z}^n \to A$.

Definition 5.2. An *Associative algebra* over a field k is a unital vector space A over k with associative bilinear operation $a, b \mapsto ab$ for $a, b \in A$.

Definition 5.3. A *Representation* of an associative algebra A is a vector space V equipped with homomorphism (action) $\rho : A \to \mathrm{End}(V)$. A *Subrepresentation* of a representation V is a subspace $U \subseteq V$ which is invariant under $\rho(a)$ for all $a \in A$. If V_1, V_2 are two representations of A then $V_1 \bigoplus V_2$ is a representation of A.

Definition 5.4. A nonzero representation V of A is called *irreducible* if its only subrepresentations are 0 and V. V is called *indecomposable* if it cannot be written as the direct sum of two nonzero representations.

Proposition 5.5. Let V_1, V_2 two representations of an algebra A over a field. Now let $\phi : V_1 \to V_2$ be a nonzero homomorphism. Then:

(1) If V_1 is irreducible, ϕ is injective.
(2) if V_2 is irreducible, ϕ is surjective.

Definition 5.6. A *Tensor Algebra* TV of a vectorspace V is defined as $TV := \bigoplus_{n \ge 0} V^{\otimes n}$ with operation $a \cdot b = a \otimes b$.

Definition 5.7. (1) The *Symmetric Algebra* $\mathrm{Sym}(V)$ of a vectorspace V is $TV/(v \otimes w - w \otimes v)$ for $v, w \in V$.
(2) The *Exterior Algebra* $\wedge V$ of a vectorspace V is $TV/(v \otimes v)$ for $v \in V$.

Definition 5.8. Let R be a ring and G a group. A *group ring* $R[G]$ or simply RG is an R-module generated by the elements of G and consists of all finite formal R-linear combinations of elements of G, with multiplication defined by extending the group operation in G.

Example 5.9. Let $R = \mathbb{C}$ and $G = C_3$. Then

$$\mathbb{C}C_3 = \left\{ z_1 1_G + z_2 a + z_3 a^2 : z_i \in \mathbb{C} \right\}$$

with addition

$$(z_1 1_G + z_2 a + z_3 a^2) + (w_1 1_G + w_2 a + w_3 a^2) = (z_1 + w_1) 1_G + (z_2 + w_2) a + (z_3 + w_3) a^2$$

and multiplication

$$(z_1 1_G + z_2 a + z_3 a^2)(w_1 1_G + w_2 a + w_3 a^2) = (z_1 w_1 + z_2 w_3 + z_3 w_2) 1_G + (z_1 w_2 + z_2 w_1 + z_3 w_3) a + (z_1 w_3 + z_2 w_2 + z_3 w_1) a^2$$

Definition 5.10. For a finite group G, let U be a $\mathbb{C}G$-module. Let $[g]$: $U \to U$ such that $u \mapsto gu$. The *character* of U is the map

$$\chi_U : G \to \mathbb{C}$$

such that

$$\chi_U(g) = tr([g]),$$

the trace of $[g]$ when represented in matrix form. Note that if ρ: $G \to GL(U)$ is the representation of G corresponding to U, then

$$\chi_U(g) = tr(\rho(g)).$$

For example, $\chi_U(1) = \dim_{\mathbb{C}} U$.

The character of U has some very interesting properties. Let $g \in G$ be of order n, then

- $\rho(g)$ is diagonalizable
- $\chi_U(g)$ is the sum (with multiplicities) of eigenvalues of $\rho(g)$
- $\chi_U(g)$ is the sum of $\chi_U(1)$ n^{th} roots of unity
- $\chi_U(g^{-1}) = \overline{\chi_U(g)}$, the complex conjugate of $\chi_U(g)$
- $\mid \chi_U(g) \mid \leq \chi_U(1)$
- $\{x \in G : \chi_U(x) = \chi_U(1)\}$ is a normal subgroup of G

Definition 5.11. If $\lambda = (\lambda_1, \lambda_2, ..., \lambda_l)$ is such that $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_l$ and $\sum_{i=1}^{l} \lambda_i = n$ then λ is a *partition* of n, written $\lambda \vdash n$.

Definition 5.12. Suppose $\lambda \vdash n$. The *Ferrers diagram*, or *shape* of λ is an array of n dots into l left-justified rows with row i containing λ_i dots. For example, the partition (3, 3, 2, 1) has Ferrers diagram

• • •
• • •
• •
•

where the dot in row i and column j has coordinates $(i,\ j)$.

Definition 5.13. Let $\lambda = (\lambda_1, ..., \lambda_l) \vdash n$. Then the corresponding *Young subgroup* of S_n is
$S_\lambda = S_{\{1,2,...,\lambda_1\}} \times S_{\{\lambda_1+1,\lambda_1+2,...,\lambda_1+\lambda_2\}} \times ... \times S_{\{n-\lambda_1+1,n-\lambda_1+2,...,n\}}$.

Example 5.14. $S_{(3,3,2,1)} = S_{\{1,2,3\}} \times S_{\{4,5,6\}} \times S_{\{7,8\}} \times S_{\{9\}} \cong S_3 \times S_3 \times S_2 \times S_1$.

In general, $S_{(\lambda_1,...,\lambda_l)}$ and $S_{\lambda_1} \times ... \times S_{\lambda_l}$ are isomorphic as groups.

Definition 5.15. Suppose $\lambda \vdash n$. A *Young tableau, t, of shape* λ is an array obtained by replacing the dots of the Ferrers diagram of λ with the numbers 1, 2, ..., n bijectively. There are $n!$ many Young tableaux for any shape $\lambda \vdash n$, one for each permutation in S_n.

Let $t_{i,j}$ stand for the entry of t in position $(i,\ j)$. A Young tableau of shape λ is also called a $\lambda - tableau$, denoted by t^λ. Alternatively, one can write sh $t = \lambda$, or the *shape* of t is that of λ.

Definition 5.16. Two λ-tableaux t_1 and t_2 are *row equivalent*, $t_1 \sim t_2$, if corresponding rows of the two tableaux contain the same elements. A *tabloid of shape* λ, or λ-*tabloid* is then

$$[t] = \{t_1 : t_1 \sim t\}$$

where sh $t = \lambda$. If $\lambda = (\lambda_1, ..., \lambda_l) \vdash n$, then the number of tableaux in any given equivalence class is $\lambda_1!\lambda_2!...\lambda_l! =: \lambda!$. Then the number of λ-tabloids is $n!/\lambda!$.

Example 5.17. The tableaux

$$\begin{matrix} 1 & 2 \\ 3 & \end{matrix} \sim \begin{matrix} 2 & 1 \\ 3 & \end{matrix}$$

are row-equivalent and their corresponding tabloid can be written

$$\begin{array}{c} \hline 1 \ \ 2 \\ \hline 3 \quad \\ \hline \end{array}$$

with horizontal lines indicating that it is a tabloid and thus that it is a representative of an equivalence class where the other row-equivalent permutations of the representative tableau are in.

Tableau t is acted upon by permutation $\pi \in S_n$ by $\pi t = (\pi(t_{(i,j)})$. Note that we will use cycle notation to describe permutations. For example

$$(1\ 2\ 3)\ \begin{matrix} 1 & 2 \\ 3 & \end{matrix} = \begin{matrix} 2 & 3 \\ 1 & \end{matrix}$$

Theorem 5.18. *The action of S_n on the tabloids by $\pi[t] = [\pi t]$ is well-defined.*

Proof. For this proof we must show that for all $\pi \in S_n$, $t \sim t' \Rightarrow \pi t \sim \pi t'$. We use induction on the two-cycle decomposition of π. First let $\sigma \in S_n$ be a two-cycle, i.e. σ switches two elements on the tableaux t and t'. That is to say, $\sigma(t_{(a,b)}) = t_{(c,d)}$, $\sigma(t_{(c,d)}) = t_{(a,b)}$, and $\sigma(t_{(i,j)}) = t_{(i,j)}$ for all $(i,j) \neq (a,b), (c,d)$. Similarly for $\sigma(t')$. Now if $t_{(a,b)} = t'_{(a,b')}$ then $\sigma(t_{(a,b)}) = t_{(c,d)} = t'_{(c,d')} = \sigma(t'_{(a,b')})$. $\sigma(t'_{(a,b')})$ must be in the c^{th} row of t' because it is in the c^{th} row of t and $t \sim t'$. Therefore for two-cycle σ, $t \sim t' \Rightarrow \sigma t \sim \sigma t'$. Now $\forall \pi \in S_n$ $\pi = \Pi_{i=1}^k \sigma_i$ such that σ_i is a two-cycle. Let the hypothesis be true for the product of k two-cycles. That is, $t \sim t' \Rightarrow \Pi_{i=1}^k \sigma_i t \sim \Pi_{i=1}^k \sigma_i t'$. To prove for $k+1$, we have $\Pi_{i=1}^{k+1} \sigma_i t = \sigma_{k+1} \Pi_{i=1}^k \sigma_i t$ and $\Pi_{i=1}^k \sigma_i t \sim \Pi_{i=1}^k \sigma_i t'$ therefore by the induction hypothesis, $\sigma_{k+1} \Pi_{i=1}^k \sigma_i t \sim \sigma_{k+1} \Pi_{i=1}^k \sigma_i t'$ and therefore $\forall \pi \in S_n$ $t \sim t' \Rightarrow \pi t \sim \pi t'$. □

Definition 5.19. Let $\lambda \vdash n$. Define $M^\lambda := \mathbb{C}[[t_1], ..., [t_k]]$ where $[t_1], ..., [t_k]$ is the complete list of λ-tabloids. M^λ is called the *permutation module corresponding to* λ. It is a $\mathbb{C}S_n$-module.

Example 5.20. If $\lambda = (n)$, then $M^{(n)} = \mathbb{C}[\ \overline{\underline{12...n}}\ ]$ is the trivial $\mathbb{C}$-module.

Example 5.21. Let $\lambda = (1, 1, ..., 1)$ n times, or $\lambda = (1^n)$. Then each equivalence class $[t]$ consists of a single tableau, identified with a permutation in S_n. Then $M^{(1^n)} \cong \mathbb{C}[S_n]$.

Example 5.22. Let $\lambda = (n-1, 1)$. Each tabloid is uniquely determined by the single element in the second row, which can be an integer from 1 to n. Therefore $M^{(n-1,1)} \cong \mathbb{C}[1, 2, ..., n]$

Definition 5.23. Let t be a tableau with rows labeled $R_1, R_2, ..., R_l$ and columns labeled $C_1, C_2, ..., C_k$. Then

$$R_t = S_{R_1} \times S_{R_2} \times ... \times S_{R_l}$$

and

$$C_t = S_{C_1} \times S_{C_2} \times ... \times S_{C_l}$$

are the *row-stabilizer* and *column-stabilizer* of t, respectively.

Note that the tabloid equivalence classes can be expressed as $[t] = R_t t$. Given a subset $H \subseteq S_n$, we can form

$$H^+ = \sum_{\pi \in H} \pi$$

and

$$H^- = \sum_{\pi \in H} sgn(\pi)\pi$$

in $\mathbb{C}S_n$. We can then define

$$\kappa_t := C_t^- = \sum_{\pi \in C_t} sgn(\pi)\pi$$

Definition 5.24. If t is a tableau, then the associated *polytabloid* is

$$e_t = \kappa_t[t]$$

and for any partition $\lambda \vdash n$, the *Specht module*, S^λ, is the submodule of M^λ spanned by the polytabloids e_t.

Definition 5.25. A tableau t is *standard* if the rows and columns of t are increasing sequences. For example

$$t = \begin{array}{lll} 1 & 2 & 3 \\ 4 & 6 & \\ 5 & & \end{array}$$

is standard, but

$$t = \begin{array}{lll} 1 & 2 & 3 \\ 5 & 4 & \\ 6 & & \end{array}$$

is not.

The set $\{e_t : \text{t is a standard } \lambda\text{-tableau}\}$ is a basis for S^λ (proof is long, see Sagan).

The goal is to connect Schur functors to multigraded Betti numbers and thereby use representation theory to contextualize our findings.

Definition 5.26. Let R be a commutative ring, E and M R-modules and $\lambda \vdash n$. Let t be a Young tableau of shape λ. Index the n-fold direct product $E \times E \times ... \times E$ with the cells of t. A *Schur functor* is a map $\phi : E^{\times n} \to M$ such that

(1) ϕ is multilinear
(2) ϕ is alternating in the entries indexed by the columns of t
(3) ϕ satisfies an exchange condition that if $I \subset \{1, 2, ..., n\}$ are numbers of column i of t then

$$\phi(x) = \sum_{x'} \phi(x')$$

where x' is an n-tuple obtained from x by exchanging the elements indexed by I with any $\mid I \mid$ elements indexed by the numbers in column $i - 1$ in order.

The Schur functor is a functor from the category of modules over a commutative ring to itself and is indexed by partitions.

Example 5.27. For an example of the exchange condition (3) let λ = (2,2,1) and

$$t = \begin{array}{ll} 1 & 4 \\ 2 & 5 \\ 3 & \end{array}$$

Let $I = \{4, 5\}$, or the entries in the second column, then we get

$$\phi(x_1, x_2, x_3, x_4, x_5) = \phi(x_4, x_5, x_3, x_1, x_2) + \phi(x_4, x_2, x_5, x_1, x_3) + \phi(x_1, x_4, x_5, x_2, x_3)$$

and taking $I = \{5\}$ gives

$$\phi(x_1, x_2, x_3, x_4, x_5) = \phi(x_5, x_2, x_3, x_4, x_1) + \phi(x_1, x_5, x_3, x_4, x_2) + \phi(x_1, x_2, x_5, x_4, x_3)$$

Let I_i be the i^{th} column of t, I_{i-1} be the $(i-1)^{th}$ and $I \subset I_i$. Then in general this equivalence will have $\binom{I_{i-1}}{I}$ summands.

Definition 5.28. Let $\lambda \vdash n$ have a Ferrers diagram L. For each cell (i, j) in L, the *hook* $H_\lambda(i, j)$ of (i, j) is the set of cells (a, b) such that $a = i$ and $b \geq j$ or $a \geq i$ and $b = j$. The *hook-length* $h_\lambda(i, j)$ is the number of cells in the hook $H_\lambda(i, j)$

The hook-length formula expresses the number of standard λ-tableaux as d_λ as

$$d_\lambda = \frac{n!}{\prod h_\lambda(i, j)}$$

where the product is over all the cells of L. Each partition $\lambda \vdash n$ corresponds to an irreducible representation of S_n and the degree of each representation is the number of standard tableaux in the shape of its corresponding partition. In other words, let Y be an irreducible representation of S_n and λ its corresponding partition. Then $\deg Y = d_\lambda$.

How does a partition $\lambda \vdash n$ correspond to an irreducible representation of S_n? First, it may be helpful to understand how partitions correspond with conjugacy classes of the symmetric group. For any symmetric group, the *cycle* of a permutation determines that permutation's conjugacy class. In other words, two permutations are conjugates iff they have the same number of cycles of each size.

Let $\lambda = (\lambda_1, ..., \lambda_l) \vdash n$ such that the first k_1 entries are equal, then the next k_2 are equal and so on. Then λ corresponds to the conjugacy class containing permutations composed of k_1 disjoint λ_1-cycles, k_2 disjoint λ_{k_1+1}-cycles, k_3 disjoint $\lambda_{k_1+k_2+1}$-cycles, and so on, or k_i disjoint $\lambda_{k_1+k_2+...+k_i+1}$-cycles.

Example 5.29. For $n = 4$, we have the following partitions $\lambda \vdash 4$ and their corresponding conjugacy classes in S_4

Partition	Conjugacy Class
(1,1,1,1)	() identity
(2,1,1)	(1,2), (1,3), (1,4), (2,3), (2,4), (3,4)
(2,2)	(1,2)(3,4), (1,3)(2,4), (1,4)(2,3)
(3,1)	(1,2,3), (1,3,2), (1,2,4), (1,4,2), (1,3,4), (1,4,3), (2,3,4), (2,4,3)
(4)	(1,2,3,4), (1,2,4,3), (1,3,2,4), (1,3,4,2), (1,4,2,3), (1,4,3,2)

This comprises all of the conjugacy classes of S_4.

Since each conjugacy class corresponds of S_n with an irreducible representation, so must each partition $\lambda \vdash n$.

We want to use this information to generalize Theorem 3.3 of the Reiner and Roberts paper. This theorem demonstrates the isomorphism between the direct sum of weight spaces $\bigoplus_\lambda V_\gamma^\lambda$ and $\bigoplus_{(\lambda,\mu)}(V^\lambda \otimes W^\mu)_{(\gamma,\delta)}$ and the reduced homologies $\tilde{H}_\bullet(\Delta_\gamma; k)$ and $\tilde{H}_\bullet(\Delta_{\gamma,\delta}; k)$ of the complete graph matching complex Δ_γ and the chessboard complex $\Delta_{\gamma,\delta}$, respectively. If these isomorphisms could be generalized for arbitrarily many partitions, or at least for three, then it will become much easier to understand the homologies given by the tripartite graphs. This requires a greater understanding of weight spaces.

Recall from the Reiner and Roberts paper that

$$\mathrm{Segre}(m,n,r) = \bigoplus_{a,b\geq 0, a=b+r} \mathrm{Sym}^a V \otimes \mathrm{Sym}^b W.$$

We would like to iterate on this equation to find some subalgebra $X(m,n,s,r)$ of $k[\mathbf{x},\mathbf{y},\mathbf{z}]$ that corresponds to a projective embedding similar to the Segre embedding

$$\sigma : \mathbb{P}^m \times \mathbb{P}^n \hookrightarrow \mathbb{P}^{(m+1)(n+1)-1}$$

such that

$$\sigma([X_0 : X_1 : ... : X_m], [Y_0 : Y_1 : ... : Y_n]) = [X_0Y_0 : X_0Y_1 : ... : X_iY_j : ... : X_mY_n].$$

Since the Segre subalgebra is used in the proof of Theorem 3.3 of the Reiner and Roberts paper, iterating upon it to accommodate three symmetric algebras could be invaluable to our investigation of tripartite graphs. Iterating on this yields an embedding

$$\delta : \mathbb{P}^m \times \mathbb{P}^n \times \mathbb{P}^s \hookrightarrow \mathbb{P}^{(m+1)(n+1)(s+1)-1}$$

which we may use to better understand the subalgebra of $k[\mathbf{x},\mathbf{y},\mathbf{z}]$ and to better understand the homology given by tripartite graphs.

6. Taylor Complex

The Taylor complex is a (in general non-minimal) resolution of a monomial ideal. The complex is a natural generalization of the Koszul complex.

Definition 6.1. [8, Excercise 17.11] Let $S = k[x_1,\ldots,x_n]$ where k is any ring. Let $m_1,\ldots,m_r$ be monomials in terms of the x_j's (that is $m_i = x_1^{p_1}x_2^{p_2}\ldots x_n^{p_n}$ for some $p_1,\ldots,p_n$). The *Taylor Complex*, denoted $T(m_1,\ldots,m_r)$, is the resolution

$$T(m_1,\ldots,m_r) : 0 \to F_r \xrightarrow{d_r} \cdots \xrightarrow{d_1} F_0$$

Where F_s is the free module on basis elements e_I where $I \subseteq \{1,\ldots,r\}$. Set $m_I = \mathrm{lcm}\{m_i \mid i \in I\}$. For each pair of subsets I,J where $|I| = s$ and $|J| = s-1$, let $I = \{i_1,\ldots,i_s\}$ and suppose that $i_1 < i_2 < \ldots < i_s$. Now define:

$$C_{I,J} = \begin{cases} 0 & \text{if } J \not\subset I \\ (-1)^k \frac{m_I}{m_J} & \text{if } I = J \cup \{i_k\} \text{ for some } k \end{cases}$$

and

$$d_s : F_s \to F_{s-1} \text{ by } e_I \mapsto \sum_J C_{I,J} e_J$$

To compute our data we will use Macaulay 2 to compute specific Segre ideals. Since we are studying Segre ideals, which are toric ideals, not monomial ideals, we will use the initial terms of every term of a Gröbner basis as generators for our monomial ideal.

Here we give the length of the Taylor Resolution for various Segre embeddings (with two parts, n_1 and n_2 not the sub-algebra discussed at the end of section 4). Since the process is very memory intensive we are not able to calculate larger n values than those listed in the table.

n_1/n_2	1	2	3	4
1	1	3	6	10
2	-	9	-	-

Due to the memory requirements and the immediate extreme gap between the length of the actual resolution compared to the taylor resolution we will not pursue this approach further.

7. Bounded Degree Graph Complexes and Homology for specific cases of Bounded Degree Graph Complexes

We introduce a new complex and a new type of simplicial complex.

Definition 7.1. A *Complete Graph with Loops on n vertices* is the graph constructed from the complete graph on n vertices by adding a loop edge at each vertex.

Definition 7.2. Let $G = (V, E)$ be a graph and $n = |V|$. Furthermore label the vertices of V form 1 to n and let $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n)$ with $\lambda_i \in \mathbb{N}$ for all $0 < i \leq n$. A *bounded degree graph complex* of a graph G with a bound λ is the simplicial complex whose faces are edge sets of subgraphs of G whose vertex degree at vertex i is less then or equal to λ_i. We label these graphs $BD(G, \lambda)$

In other words we have a list of bounds on the vertex that the subgraph must meet. This bound requirement is closed under edge deletion so this is indeed a simplicial complex. The case of $\lambda_i = 1$ for $0 < i \leq n$ is the matching complex since a subgraph is a matching if and only if each vertex has degree at most 1. This is also a generalization of Definition 4.3, we can use a bipartite graph for G with vertex partitions V_1 and V_2 and label the vertical edge of the chessboard with elements of V_1 and the horizontal edge with elements of V_2. This will give us the same complex.

The notation used in [1] and other sources for the bounded degree complex for complete graphs and complete graphs with loops on n vertices are BD_n^λ and $\overline{BD_n^\lambda}$ respectively. We avoid this since we will consider various other graphs. Furthermore we may choose to split λ, i.e. given $\lambda = (\lambda_1, ..., \lambda_k, \lambda_{k+1}, ..., \lambda_n)$, $BD(G, \lambda) = BD(G, (\lambda_1, ..., \lambda_k), (\lambda_{k+1}, ...\lambda_n))$. We will do this when it is still clear what the vertex assignment of the bounds are.

We provide the proof of homologies or homotopy equivalences for specific cases of the various bounded degree graphs in Sections 8 ,9, 10 ,11.

For bounded degree complete graphs because of the symmetry of the underlying complete graph, we are able to replace any vertex in the underlying graph with another vertex. This means that we given a degree bound, any permutation of the bounds will yield isomorphic spaces. This will allow us to reduce the number of computations needed. In Sections 8 ,9, 10 ,11, increasing sequences were used to index different sets of bounds. We did this because there was more apparent pattern in the bottom non-vanishing homology when we ordered them lexicographically in this indexing. In this section we use weakly decreasing sequences or partitions because they are slightly easier to work with.

The degree of a vertex of complete graphs on n vertices's is $n - 1$, so the maximum values for the bounds λ_i we need to consider is $n - 1$.

Theorem 7.3. *For bounded degree complex of a complete graph on n vertices, $BD(K_n, \lambda)$, and suppose λ is of or is a permutation of partition $(n-1, n-1, \lambda_3, ..., \lambda_n)$. Then $BD(K_n, \lambda)$ is homotopy equivalent to a point. Thus the homology is trivial everywhere and there is no torsion.*

Proof. Consider $BD(K_n, (n-1, n-1, \lambda_3, ..., \lambda_n))$, all other cases are isomorphic. The vertices 1 and 2 have the maximum bounds $n-1$. So given a subgraph that meets the bound requirements it either contain the edge 12 or can be extended to include 12. Thus the simplex is a cone described in Definition 3.3 and is homotopy equivalent to a point by 3.4. □

We can generalize this, the proof is similar to the one for Theorem 7.3

Theorem 7.4. *Given a graph G (possibly with loops) and a bound $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n)$ such that for some vertex i and j (possibly non-distinct) connected by an edge, λ_i and λ_j are equal to the degree of vertex i and j respectively, $BD(G, \lambda)$ is homotopy equivalent to a point.*

8. Data for Bounded Degree Complete Graphs

Below is some data of the homology of several bounded degree graph complexes. The rational homology for some of the bounded degree graph complexes are known as a result of Proposition

4.4. We will be looking for any torsion present in those cases. The code we used is given in Section 13.
The following is the data for the complete graph with bounded degree(multiplicity). These are not the ones that correspond to Veronese Ideals, those are given later in the section(to be added). The ones listed below are the non-vanishing ones, other bounds give us vanishing homology. For complete graphs because of the symmetry we only need to consider weakly increasing sequences for the bounds and the maximum bound we need to consider is the *number of vertices*−1. All other bounds can be achieved by relabeling the vertices.

Complete Graph on 4 vertices, non-vanishing homology bounds

Partition/Homology	0	1	2	3	Euler
(1,1,1,1)	$\mathbb{Z}^2$	0	-	-	3
(1,1,1,2)	0	$\mathbb{Z}$	-	-	0
(1,1,2,2)	0	$\mathbb{Z}^2$	0	-	-1
(1,1,2,3)	0	$\mathbb{Z}$	0	-	0
(2,2,2,2)	0	0	$\mathbb{Z}^3$	0	4
(2,2,2,3)	0	0	$\mathbb{Z}$	0	2

Complete Graph on 5 vertices, non-vanishing homology bounds

Partition/Homology	0	1	2	3	4	5	6	7	Euler
(1,1,1,1,1)	0	$\mathbb{Z}^6$	-	-	-	-	-	-	-5
(1,1,1,1,2)	0	$\mathbb{Z}^6$	-	-	-	-	-	-	-5
(1,1,1,1,3)	0	$\mathbb{Z}^3$	$\mathbb{Z}$	0	-	-	-	-	-1
(1,1,1,1,4)	0	$\mathbb{Z}^3$	0	0	-	-	-	-	-2
(1,1,1,3,3)	0	0	$\mathbb{Z}^2$	0	-	-	-	-	3
(1,1,1,3,4)	0	0	$\mathbb{Z}$	0	-	-	-	-	2
(1,1,2,2,2)	0	0	$\mathbb{Z}^6$	0	-	-	-	-	7
(1,1,2,2,3)	0	0	$\mathbb{Z}^2$	0	-	-	-	-	3
(1,1,2,2,4)	0	0	$\mathbb{Z}^2$	0	-	-	-	-	3
(1,2,2,2,2)	0	0	$\mathbb{Z}^3$	0	-	-	-	-	4
(1,2,2,2,3)	0	0	0	$\mathbb{Z}^2$	0	-	-	-	-1
(1,2,2,3,3)	0	0	0	$\mathbb{Z}^2$	0	-	-	-	-1
(1,2,2,3,4)	0	0	0	$\mathbb{Z}$	0	-	-	-	0
(2,2,2,2,2)	0	0	0	$\mathbb{Z}^9$	0	-	-	-	-8
(2,2,2,2,3)	0	0	0	$\mathbb{Z}^5$	0	-	-	-	-4
(2,2,2,2,4)	0	0	0	$\mathbb{Z}^3$	0	-	-	-	-2
(2,2,3,3,3)	0	0	0	0	$\mathbb{Z}^3$	0	-	-	4
(2,2,3,3,4)	0	0	0	0	$\mathbb{Z}^3$	0	-	-	4
(3,3,3,3,3)	0	0	0	0	0	$\mathbb{Z}^4$	0	-	-5
(3,3,3,3,4)	0	0	0	0	0	$\mathbb{Z}$	0	0	0

More Euler Characteristic Data for various $(1,\dots 1)$ partition complete graphs

Complete graph vertices	Euler Characteristic
2	1
3	3
4	3
5	-5
6	-15
7	21
8	133
9	-27
10	-1215
11	-935
12	12441

9. Data for Bounded Degree Complete Graphs with loops

These correspond to the Veronese Ideals from Proposition 4.4 and the code adjusted for loops is in Section 13. We look back at the simplicial complex defined in Section 4. These bounded degree subgraphs of complete graphs with loops corresponds to Veronese subalgebras. If the multidegree has all ones entries then the bounded degree complexes for complete graphs and complete graphs with loops are isomorphic since loops contribute two degrees to a vertex. The weight spaces of $Tor^{A_n}(Veronese(n,2,0),k)$ correspond to the bounded degree complexes of a complete graph with loops and n vertices's whose components of the multidegree sum up to an even number even. Likewise the weight spaces of $Tor_i^{A_n}(Veronese(n,2,1),k)$ correspond to the multidegrees with an odd sum. For example the bounded degree complete graph with loops on 5 vertices with multidegree (1,1,1,1,1) will correspond to a weight spaces of $Tor_i^{A_n}(Veronese(5,2,1),k)$.

Below is the data for bounded degree complexes of complete graphs on 3 and 4 vertices. We furthermore have data for 5 vertices which is omitted because of the size. We use weakly increasing sequences for the multidegrees. Any multidegree obtained from a permutation of the values in an increasing sequences will yield an isomorphic complex. All complexes with trivial homology are omitted. Below in the tables we see that for both $n=3$ and $n=4$, the homology groups of any of the bounded degree complexes have torsion. This is also true for n=5, the homology groups are all free. This tells us that $Tor_i^{A_n}(Veronese(n,2,r),k)$ is not dependent on characteristic for $n<6$. We are still computing the case of $n=6$, we have about 1/6 of all possible bounded degree complexes for $n=6$ computed and we have not found any torsion yet.

For the case of $n=7$ we know that the matching complex of the complete graph on 7 vertices is $\mathbb{Z}/3\mathbb{Z}$, so this tells us that the resolution for $Veronese(7,2,1)$ is at least dependent on whether we work in characteristic 3 or not. It is also mentioned by Reiner and Roberts [5] that it has been proven by direct computation of homology of the bounded chessboard complex for $n=7$ and multidegree $(2,2,2,2,2,2,2)$ that $Tor_5^{A_7}(Veronese(n,2,0),k)$ is dependent on if the field is characteristic 5.

Complete Graph with loops on 3 vertices, homology

Partition/Homology	0	1	2	3
(1, 1, 1)	$\mathbb{Z}^2$	0	-	-
(1, 1, 2)	0	$\mathbb{Z}^3$	-	-
(1, 1, 3)	0	$\mathbb{Z}^2$	0	-
(1, 2, 2)	0	0	$\mathbb{Z}$	-
(1, 2, 3)	0	$\mathbb{Z}^4$	0	-
(2, 2, 2)	0	0	$\mathbb{Z}$	-
(2, 3, 3)	0	0	$\mathbb{Z}$	0

Complete Graph with loops on 4 vertices, homology

Partition/Homology	0	1	2	3	4	5	6	7	Euler
(1, 1, 1, 1)	$\mathbb{Z}^2$	0	-	-	-	-	-	-	3
(1, 1, 1, 2)	0	$\mathbb{Z}^3$	-	-	-	-	-	-	-2
(1, 1, 1, 3)	0	$\mathbb{Z}^2$	0	-	-	-	-	-	-1
(1, 1, 1, 4)	0	0	$\mathbb{Z}$	-	-	-	-	-	2
(1, 1, 2, 2)	0	$\mathbb{Z}^4$	0	-	-	-	-	-	-3
(1, 1, 2, 3)	0	0	$\mathbb{Z}$	-	-	-	-	-	2
(1, 1, 2, 4)	0	0	$\mathbb{Z}$	0	-	-	-	-	2
(1, 1, 3, 3)	0	0	$\mathbb{Z}^2$	0	-	-	-	-	3
(1, 2, 2, 2)	0	0	$\mathbb{Z}$	-	-	-	-	-	2
(1, 2, 2, 3)	0	0	$\mathbb{Z}^4$	0	-	-	-	-	5
(1, 2, 3, 3)	0	0	$\mathbb{Z}$	0	-	-	-	-	0
(1, 2, 3, 4)	0	0	0	$\mathbb{Z}$	0	-	-	-	0
(1, 3, 3, 3)	0	0	0	$\mathbb{Z}^2$	0	-	-	-	-1
(1, 3, 3, 4)	0	0	0	$\mathbb{Z}$	0	-	-	-	0
(2, 2, 2, 2)	0	0	$\mathbb{Z}^6$	0	-	-	-	-	7
(2, 2, 2, 3)	0	0	$\mathbb{Z}$	0	-	-	-	-	2
(2, 2, 2, 4)	0	0	0	$\mathbb{Z}^2$	0	-	-	-	-1
(2, 2, 3, 3)	0	0	0	$\mathbb{Z}^4$	0	-	-	-	-3
(2, 3, 3, 3)	0	0	0	$\mathbb{Z}^3$	0	-	-	-	-2
(3, 3, 3, 3)	0	0	0	0	$\mathbb{Z}^2$	0	-	-	3
(3, 3, 3, 4)	0	0	0	0	$\mathbb{Z}^2$	0	-	-	3

The relationship between the growth of the homology and the partition may be correlated to the increase of the number of tabloids for each partition. This number is given by the formula in 4.17. As the number of possible tabloids for each partition increases, so too does the minimal homology given by that partition. Therefore there may exist bounds indicating the minimal homology given by a partition.

Another possibility we've considered was that the minimal homology of a partition was given by the number of standard tableaux given by that partition's shape. This is given by the hook-length formula in 4.29. This however does not seem to be the case, because there are more standard tableaux given by (1,1,1,1,4) than (1,1,2,2,2) but the latter has a greater minimal homology than the former.

The correlation between the minimal nontrivial homology and the number of tabloids given by the corresponding partitions seems to still hold under the pairs.

For valuse of $n > 6$ we recall the following table for matching complexes form earlier,

n	$\tilde{H}_{\nu_v}(M_n)$
7	$\mathbb{Z}/3\mathbb{Z}$
8	$\mathbb{Z}^{132}$
9	$\mathbb{Z}^{42} \bigoplus (\mathbb{Z}/3\mathbb{Z})^8$
10	$\mathbb{Z}/3\mathbb{Z}$
11	$\mathbb{Z}^{1188} \bigoplus (\mathbb{Z}/3\mathbb{Z})^{45}$
12	$(\mathbb{Z}/3\mathbb{Z})^{56}$
13	$\mathbb{Z}/3\mathbb{Z}$

Furthermore there is a theorem in Shareshian and Wachs [6]

Theorem 9.1. [6, Theorem 1.3] *For* $n > 3$, $v_n = \lfloor \frac{n+1}{3} \rfloor - 1$ $H_{v_n}(MK_n)$ *is finite iff* $n \geq 7$ *and* $n \notin \{8, 9, 11\}$.

From the table we see that for $n = 9, 11$ the bottom nonvanishing homologies contain a copy of $\mathbb{Z}/3\mathbb{Z}$. This tells us for $n \geq 9$ and n even that $T^{A_n}_{v_n+1}(Veronese(n, 2, 0), k)$ is dependent on the characteristic of k. Furthermore for $n \geq 7$ and n odd that $T^{A_n}_{v_n+1}(Veronese(n, 2, 1), k)$ is dependent on the characteristic of k.
We can run some calculation for higher n values given the multidegree contains small numbers. The result(s) below are the homology results that are not free.

Select Homology for Bounded Degree Complete Graphs With Loops on 8 Vertices's

multidegree/homology	0	1	2	3
(2,1,1,1,1,1,1,1)	0	0	$\mathbb{Z}^{21} \oplus \mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}^{35}$

The above tells us that $Tor^{A_8}_3(Veronese(8, 2, 1), k)$ is dependent on the character of k. We have not been able to find any characteristic dependence for $Tor^{A_8}_i(Veronese(8, 2, 0), k)$.

10. Data for Chessboard Complex

We have collected the data for various different sized Chessboard Complexes (i.e. the data corresponding to a the complex created from a bipartite graph), the complete data for the $(2, 2), (2, 3), (3, 3)$ and partial data for the $(3, 4)$ and $(5, 5)$ cases are included in this report but we have similar data for $(4, 4)$ and $(2, 4)$. For each case we have also computed the associated betti numbers.

The bounded degree chessboard complexes correspond to Segre subalgebras. Again going back to Section 4, We look at the complex $K_{(\mu,\delta)}$ defined there. $K_{(\mu,\delta)}$ will be non-empty when we let the module M from Section 4 be Segre(n, m, r), where $r = \sum_i^n \mu_i - \sum_j^m \delta_j$. In such case $K_{(\mu,\delta)}$ will be isomorphic to the bounded degree chessboard complex with multiplicities μ and δ. Thus the bounded degree chessboard complexes on (n, m) vertices correspond to weight spaces of $Tor^{A_{m,n}}_i(Segre(n, m, r), k)$, where $r = \sum_i^n \mu_i - \sum_j^m \delta_j$.

From the tables below and data we have omitted we show that $Tor^{A_{m,n}}_i(Segre(n, m, r), k)$ is not dependent on the characteristic of the field k when $n \leq 4$ and $m \leq 4$.

Chessboard Complex with $m = 2, n = 2$ partition non-vanishing homology bounds

Partition/Homology	0	1	2	Euler
$(1,1),(1,1)$	$\mathbb{Z}$	0	-	2
$(1,1),(2,2)$	0	$\mathbb{Z}$	0	0

Chessboard Complex with $m = 2, n = 3$ partition non-vanishing homology bounds

Partition/Homology	0	1	2	3	Euler
$(1,1),(1,1,1)$	0	$\mathbb{Z}$	-	-	0
$(1,1),(1,1,2)$	0	$\mathbb{Z}^2$	-	-	-1
$(1,1),(1,2,2)$	0	$\mathbb{Z}^3$	-	-	-2
$(1,1),(2,2,2)$	0	$\mathbb{Z}^4$	-	-	-3
(1,2),(1,1,1)	0	$\mathbb{Z}$	0	-	0
(1,2),(1,2,2)	0	0	$\mathbb{Z}$	-	2
(1,2),(2,2,2)	0	0	$\mathbb{Z}^2$	-	3
(2,2),(1,1,1)	0	$\mathbb{Z}$	0	-	0
(2,2),(2,2,2)	0	0	0	$\mathbb{Z}$	0
(3,3),(1,1,1)	0	0	$\mathbb{Z}$	-	2

Chessboard Complex with $m = 3, n = 3$ partition non-vanishing homology bounds

Partition/Homology	0	1	2	3	4	5	Euler
(1,1,1),(1,1,1)	0	$\mathbb{Z}^4$	0	-	-	-	-3
(1,1,1),(1,1,2)	0	$\mathbb{Z}$	0	-	-	-	0
(1,1,1),(1,2,2)	0	0	$\mathbb{Z}^2$	-	-	-	3
(1,1,1),(1,2,3)	0	0	$\mathbb{Z}^3$	-	-	-	4
(1,1,1),(1,3,3)	0	0	$\mathbb{Z}^4$	-	-	-	5
(1,1,1),(2,2,2)	0	0	$\mathbb{Z}^5$	-	-	-	6
(1,1,1),(2,2,3)	0	0	$\mathbb{Z}^6$	-	-	-	7
(1,1,1),(2,3,3)	0	0	$\mathbb{Z}^7$	-	-	-	8
(1,1,1),(3,3,3)	0	0	$\mathbb{Z}^8$	-	-	-	9
(1,1,2),(1,1,2)	0	0	$\mathbb{Z}$	0	-	-	2
(1,1,2),(1,2,2)	0	0	$\mathbb{Z}$	0	-	-	2
(1,1,2),(1,3,3)	0	0	0	$\mathbb{Z}$	-	-	0
(1,1,2),(2,2,2)	0	0	0	$\mathbb{Z}$	-	-	0
(1,1,2),(2,2,3)	0	0	0	$\mathbb{Z}^2$	-	-	-1
(1,1,2),(2,3,3)	0	0	0	$\mathbb{Z}^3$	-	-	-2
(1,1,2),(3,3,3)	0	0	0	$\mathbb{Z}^4$	-	-	-3
(1,2,2),(2,2,2)	0	0	0	$\mathbb{Z}$	0	-	0
(1,2,2),(2,3,3)	0	0	0	0	$\mathbb{Z}$	-	2
(1,2,2),(3,3,3)	0	0	0	0	$\mathbb{Z}^2$	-	3
(2,2,2),(2,2,2)	0	0	0	$\mathbb{Z}$	0	0	0
(2,2,2),(3,3,3)	0	0	0	0	0	$\mathbb{Z}$	0

Chessboard Complex with $m = 3, n = 4$ partition non-vanishing homology bounds

Partition/Homology	0	1	2	3	4	5	6	7	8	Euler
(1,1,1),(1,1,1,1)	0	$\mathbb{Z}^2$	$\mathbb{Z}$	-	-	-	-	-	-	0
(1,1,1),(1,1,1,2)	0	0	$\mathbb{Z}^5$	-	-	-	-	-	-	6
(1,1,1),(1,1,1,3)	0	0	$\mathbb{Z}^6$	-	-	-	-	-	-	7
(1,1,1),(1,1,2,2)	0	0	$\mathbb{Z}^{11}$	-	-	-	-	-	-	12
(1,1,1),(1,1,2,3)	0	0	$\mathbb{Z}^{12}$	-	-	-	-	-	-	13
(1,1,1),(1,1,3,3)	0	0	$\mathbb{Z}^{13}$	-	-	-	-	-	-	14
(1,1,1),(1,2,2,2)	0	0	$\mathbb{Z}^{17}$	-	-	-	-	-	-	18
(1,1,1),(1,2,2,3)	0	0	$\mathbb{Z}^{18}$	-	-	-	-	-	-	19
(1,1,1),(1,2,3,3)	0	0	$\mathbb{Z}^{19}$	-	-	-	-	-	-	20
(1,1,1),(1,3,3,3)	0	0	$\mathbb{Z}^{20}$	-	-	-	-	-	-	21
(1,1,1),(2,2,2,2)	0	0	$\mathbb{Z}^{23}$	-	-	-	-	-	-	24
(1,1,1),(2,2,2,3)	0	0	$\mathbb{Z}^{24}$	-	-	-	-	-	-	25
(1,1,1),(2,2,3,3)	0	0	$\mathbb{Z}^{25}$	-	-	-	-	-	-	26
(1,1,1),(2,3,3,3)	0	0	$\mathbb{Z}^{26}$	-	-	-	-	-	-	27
(1,1,1),(3,3,3,3)	0	0	$\mathbb{Z}^{27}$	-	-	-	-	-	-	28
(1,1,2),(1,1,1,1)	0	0	$\mathbb{Z}^5$	0	-	-	-	-	-	6
(1,1,2),(1,1,1,2)	0	0	$\mathbb{Z}^2$	0	-	-	-	-	-	3
(1,1,2),(1,1,2,2)	0	0	0	$\mathbb{Z}^3$	-	-	-	-	-	-2
(1,1,2),(1,1,2,3)	0	0	0	$\mathbb{Z}^5$	-	-	-	-	-	-4
(1,1,2),(1,1,3,3)	0	0	0	$\mathbb{Z}^7$	-	-	-	-	-	-6
(1,1,2),(1,2,2,2)	0	0	0	$\mathbb{Z}^{10}$	-	-	-	-	-	-9
(1,1,2),(1,2,2,3)	0	0	0	$\mathbb{Z}^{12}$	-	-	-	-	-	-11
(1,1,2),(1,2,3,3)	0	0	0	$\mathbb{Z}^{14}$	-	-	-	-	-	-13
(1,1,2),(1,3,3,3)	0	0	0	$\mathbb{Z}^{16}$	-	-	-	-	-	-15
(1,1,2),(2,2,2,2)	0	0	0	$\mathbb{Z}^{19}$	-	-	-	-	-	-18
(1,1,2),(2,2,2,3)	0	0	0	$\mathbb{Z}^{21}$	-	-	-	-	-	-20
(1,1,2),(2,2,3,3)	0	0	0	$\mathbb{Z}^{23}$	-	-	-	-	-	-22
(1,1,2),(2,3,3,3)	0	0	0	$\mathbb{Z}^{25}$	-	-	-	-	-	-24
(1,1,2),(3,3,3,3)	0	0	0	$\mathbb{Z}^{27}$	-	-	-	-	-	-26
(1,1,3),(1,1,1,1)	0	0	$\mathbb{Z}$	0	-	-	-	-	-	2
(1,1,3),(1,1,1,2)	0	0	0	$\mathbb{Z}$	0	-	-	-	-	0
(1,1,3),(1,1,2,2)	0	0	0	$\mathbb{Z}$	0	-	-	-	-	0
(1,1,3),(1,1,3,3)	0	0	0	0	$\mathbb{Z}$	-	-	-	-	2
(1,1,3),(1,2,2,2)	0	0	0	0	$\mathbb{Z}$	-	-	-	-	2
(1,1,3),(1,2,2,3)	0	0	0	0	$\mathbb{Z}^2$	-	-	-	-	3
(1,1,3),(1,2,3,3)	0	0	0	0	$\mathbb{Z}^3$	-	-	-	-	4
(1,1,3),(1,3,3,3)	0	0	0	0	$\mathbb{Z}^4$	-	-	-	-	5
(1,1,3),(2,2,2,2)	0	0	0	0	$\mathbb{Z}^5$	-	-	-	-	6
(1,1,3),(2,2,2,3)	0	0	0	0	$\mathbb{Z}^6$	-	-	-	-	7
(1,1,3),(2,2,3,3)	0	0	0	0	$\mathbb{Z}^7$	-	-	-	-	8

The rest of the table is omitted as there is no torsion or other immediately noticeable interesting patterns.

Euler Characteristic for various chessboard complexes

Chessboard Complex	Euler Characteristic
$M_{1,1}$	1
$M_{1,2}$	2
$M_{1,3}$	3
$M_{1,4}$	4
$M_{1,5}$	5
$M_{1,6}$	6
$M_{1,7}$	7
$M_{1,8}$	8
$M_{2,2}$	2
$M_{2,3}$	0
$M_{2,4}$	-4
$M_{2,5}$	-10
$M_{2,6}$	-18
$M_{2,7}$	-28
$M_{2,8}$	-40
$M_{3,3}$	-3
$M_{3,4}$	0
$M_{3,5}$	15
$M_{3,6}$	48
$M_{3,7}$	105
$M_{3,8}$	192
$M_{4,4}$	16
$M_{4,5}$	20
$M_{4,6}$	-36
$M_{4,7}$	-224
$M_{4,8}$	-640
$M_{5,5}$	-55
$M_{5,6}$	-150
$M_{5,7}$	35
$M_{5,8}$	1160
$M_{6,6}$	186
$M_{6,7}$	1092
$M_{6,8}$	888
$M_{7,7}$	-203
$M_{7,8}$	-7840
$M_{8,8}$	-6208

In general the formula for the Euler characteristic of a chessboard complex $M_{m,n}$, as given from the characteristic polynomial[7, Theorem 4.20], is:

$$\chi_{M_{m,n}} = \begin{cases} (-1)^m L_m^{n-m} & \text{if } n > m \\ (-1)^n L_n & \text{if } n = m \end{cases}$$

Where

$$L_n = \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{k!}, \ L_m^{n-m} = \sum_{k=0}^{n} (-1)^k \binom{2n-m}{n-k} \frac{1}{k!}$$

The following data corresponds to the homology groups for the chessboard complex over the finite field $\mathbb{Z}/3\mathbb{Z}$ (over any prime p this data is identical by replacing $\mathbb{Z}/3\mathbb{Z}$ with $\mathbb{Z}/p\mathbb{Z}$). We compute this data by applying the splitting property of the Universal coefficient theorem. We include the data for the chessboard complex with $m = 2, n = 2$ and $m = 2, n = 3$ but we have also computed the data for $m = 2, n = 4$ and $m = 3, n = 3$ and $m = 3, n = 4$.

Chessboard Complex with $m = 2, n = 2$ partition non-vanishing reduced homology groups

Partition/Homology	0	1
(1,1),(1,1)	$\mathbb{Z}/3\mathbb{Z}$	0
(1,1),(2,2)	0	$\mathbb{Z}/3\mathbb{Z}$

x

Chessboard Complex with $m = 2, n = 3$ partition non-vanishing homology groups

Partition/Homology	0	1	2	3
(1,1),(1,1,1)	0	$\mathbb{Z}/3\mathbb{Z}$	-	-
(1,1),(1,1,2)	0	$(\mathbb{Z}/3\mathbb{Z})^2$	-	-
(1,1),(1,2,2)	0	$(\mathbb{Z}/3\mathbb{Z})^3$	-	-
(1,1),(2,2,2)	0	$(\mathbb{Z}/3\mathbb{Z})^4$	-	-
(1,2),(1,1,1)	0	$\mathbb{Z}/3\mathbb{Z}$	0	-
(1,2),(1,2,2)	0	0	$\mathbb{Z}/3\mathbb{Z}$	-
(1,2),(2,2,2)	0	0	$(\mathbb{Z}/3\mathbb{Z})^2$	-
(2,2),(1,1,1)	0	$\mathbb{Z}/3\mathbb{Z}$	0	-
(2,2),(2,2,2)	0	0	0	$\mathbb{Z}/3\mathbb{Z}$
(3,3),(1,1,1)	0	0	$\mathbb{Z}/3\mathbb{Z}$	-

11. Data for Complex arising from Tripartite Graphs

We have collected the data for various different sized complexes which arise from tripartite graphs , the complete data for the $(2, 2, 2)$ case is included in this report. We also have data for the $(2, 2, 3)$, and $(2, 3, 3)$ cases as well as partial data for the $(3, 3, 3)$ case as well. For each case we have also computed the associated betti numbers.

Complex with parts $2, 2, 2$ partition non-vanishing homology bounds

Partition/Homology	0	1	2	3	4	5	6	7	8
(1,1),(1,1),(1,1)	0	$\mathbb{Z}^{11}$	0	-	-	-	-	-	-
(1,1),(1,1),(1,2)	0	$\mathbb{Z}^2$	$\mathbb{Z}$	-	-	-	-	-	-
(1,1),(1,1),(1,3)	0	$\mathbb{Z}^2$	$\mathbb{Z}$	0	-	-	-	-	-
(1,1),(1,1),(2,2)	0	$\mathbb{Z}^2$	$\mathbb{Z}^5$	0	-	-	-	-	-
(1,1),(1,1),(2,3)	0	$\mathbb{Z}^2$	$\mathbb{Z}^5$	0	-	-	-	-	-
(1,1),(1,1),(3,3)	0	$\mathbb{Z}^2$	$\mathbb{Z}^5$	0	-	-	-	-	-
(1,1),(1,2),(1,2)	0	0	$\mathbb{Z}^8$	0	-	-	-	-	-
(1,1),(1,2),(1,3)	0	0	$\mathbb{Z}^3$	0	-	-	-	-	-
(1,1),(1,2),(2,2)	0	0	$\mathbb{Z}^5$	0	-	-	-	-	-
(1,1),(1,2),(2,3)	0	0	$\mathbb{Z}^3$	0	0	-	-	-	-
(1,1),(1,2),(3,3)	0	0	$\mathbb{Z}^3$	$\mathbb{Z}^2$	0	-	-	-	-
(1,1),(1,3),(2,2)	0	0	0	$\mathbb{Z}$	0	-	-	-	-
(1,1),(1,3),(2,3)	0	0	0	$\mathbb{Z}$	0	-	-	-	-
(1,1),(1,3),(3,3)	0	0	0	$\mathbb{Z}$	0	-	-	-	-
(1,1),(2,2),(2,2)	0	0	0	$\mathbb{Z}^4$	0	-	-	-	-
(1,1),(2,2),(2,3)	0	0	0	$\mathbb{Z}^2$	0	-	-	-	-
(1,1),(2,2),(3,3)	0	0	0	$\mathbb{Z}^2$	0	0	-	-	-
(1,1),(2,2),(3,3)	0	0	0	$\mathbb{Z}^2$	0	0	-	-	-
(1,2),(1,2),(1,2)	0	0	$\mathbb{Z}^5$	0	-	-	-	-	-
(1,2),(1,2),(1,3)	0	0	$\mathbb{Z}$	$\mathbb{Z}^2$	0	-	-	-	-
(1,2),(1,2),(2,2)	0	0	$\mathbb{Z}$	$\mathbb{Z}^7$	0	-	-	-	-
(1,2),(1,2),(2,3)	0	0	$\mathbb{Z}$	$\mathbb{Z}^5$	0	-	-	-	-
(1,2),(1,2),(3,3)	0	0	$\mathbb{Z}$	$\mathbb{Z}^5$	0	0	-	-	-
(1,2),(1,3),(1,3)	0	0	0	$\mathbb{Z}^2$	0	-	-	-	-
(1,2),(1,3),(2,2)	0	0	0	$\mathbb{Z}^4$	0	-	-	-	-
(1,2),(1,3),(2,3)	0	0	0	$\mathbb{Z}$	0	0	-	-	-
(1,2),(1,3),(3,3)	0	0	0	$\mathbb{Z}$	$\mathbb{Z}$	0	-	-	-
(1,2),(2,2),(2,2)	0	0	0	$\mathbb{Z}^{10}$	0	-	-	-	-
(1,2),(2,2),(2,3)	0	0	0	$\mathbb{Z}^2$	$\mathbb{Z}$	0	-	-	-
(1,2),(2,2),(3,3)	0	0	0	$\mathbb{Z}^2$	$\mathbb{Z}^3$	0	-	-	-
(1,2),(2,3),(2,3)	0	0	0	0	$\mathbb{Z}^2$	0	-	-	-
(1,2),(2,3),(3,3)	0	0	0	0	$\mathbb{Z}$	0	0	-	-
(1,3),(1,3),(2,2)	0	0	0	0	$\mathbb{Z}$	0	-	-	-
(1,3),(1,3),(2,3)	0	0	0	0	$\mathbb{Z}$	0	-	-	-
(1,3),(1,3),(3,3)	0	0	0	0	$\mathbb{Z}$	0	0	-	-
(1,3),(2,2),(2,2)	0	0	0	$\mathbb{Z}$	$\mathbb{Z}^3$	0	-	-	-
(1,3),(2,2),(2,3)	0	0	0	0	$\mathbb{Z}^4$	0	-	-	-
(1,3),(2,2),(3,3)	0	0	0	0	$\mathbb{Z}^3$	0	0	-	-
(1,3),(2,3),(3,3)	0	0	0	0	0	$\mathbb{Z}$	0	-	-
(2,2),(2,2),(2,2)	0	0	0	$\mathbb{Z}^3$	$\mathbb{Z}^7$	0	-	-	-
(2,2),(2,2),(2,3)	0	0	0	$\mathbb{Z}$	$\mathbb{Z}^9$	0	-	-	-
(2,2),(2,2),(3,3)	0	0	0	$\mathbb{Z}$	$\mathbb{Z}^6$	$\mathbb{Z}$	0	-	-
(2,2),(2,3),(2,3)	0	0	0	0	$\mathbb{Z}^2$	0	0	-	-
(2,2),(2,3),(3,3)	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}^3$	0	-	-
(2,2),(3,3),(3,3)	0	0	0	0	0	$\mathbb{Z}^2$	0	0	-
(2,3),(2,3),(2,3)	0	0	0	0	0	$\mathbb{Z}^4$	0	-	-
(2,3),(2,3),(3,3)	0	0	0	0	0	$\mathbb{Z}^2$	0	0	-
(2,3),(3,3),(3,3)	0	0	0	0	0	0	$\mathbb{Z}^2$	0	-
(3,3),(3,3),(3,3)	0	0	0	0	0	0	$\mathbb{Z}^2$	0	0

For the $(2,2,3)$ case (for which we have the complete data, although it is not included in this report) we found torsion only in the 1st homology group for partition $(1,1),(1,1),(1,1,1)$.

Specifically:

Partition/Homology	0	1	2
(1,1),(1,1),(1,1,1)	0	$\mathbb{Z}^2 \bigoplus \mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}^3$

Similarly for the $(2,3,3)$ case we include the partitions for which we computed torsion:

Partition/Homology	0	1	2	3	4	5	6	7
(1,1),(1,1,2),(2,2,2)	0	0	0	$\mathbb{Z}^{24} \bigoplus \mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}^7$	0	-	-
(1,2),(1,1,1),(2,2,2)	0	0	$\mathbb{Z}^3$	$\mathbb{Z}^{39} \bigoplus \mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}^8$	0	-	-
(1,2),(1,1,3),(2,2,2)	0	0	0	0	$\mathbb{Z}^{12} \bigoplus \mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}^6$	0	-
(1,2),(1,2,2),(2,2,2)	0	0	0	0	$\mathbb{Z}^{26} \bigoplus (\mathbb{Z}/3\mathbb{Z})^2$	$\mathbb{Z}^{13}$	0	-
(2,2),(1,1,2),(2,2,2)	0	0	0	0	$\mathbb{Z}^{46} \bigoplus (\mathbb{Z}/3\mathbb{Z})^2$	$\mathbb{Z}^{17}$	0	-
(2,2),(1,2,2),(1,2,2)	0	0	0	0	$\mathbb{Z}^{25} \bigoplus (\mathbb{Z}/3\mathbb{Z})^5$	$\mathbb{Z}^{29}$	0	-
(2,2),(1,2,3),(2,2,2)	0	0	0	0	0	$\mathbb{Z}^{58} \bigoplus \mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}^4$	0
(2,2),(2,2,2),(2,2,2)	0	0	0	0	0	$\mathbb{Z}^{101} \bigoplus (\mathbb{Z}/3\mathbb{Z})^5$	$\mathbb{Z}^{10}$	0

And for $(2,3,3)$ over $\mathbb{Z}/3\mathbb{Z}$ we include the partitions for which we computed torsion over $\mathbb{Z}$:

Partition/Homology	0	1	2	3	4	5	6	7
(1,1),(1,1,2),(2,2,2)	0	0	0	$(\mathbb{Z}/3\mathbb{Z})^{25}$	$(\mathbb{Z}/3\mathbb{Z})^8$	0	-	-
(1,2),(1,1,1),(2,2,2)	0	0	$(\mathbb{Z}/3\mathbb{Z})^3$	$(\mathbb{Z}/3\mathbb{Z})^{40}$	$(\mathbb{Z}/3\mathbb{Z})^9$	0	-	-
(1,2),(1,1,3),(2,2,2)	0	0	0	0	$(\mathbb{Z}/3\mathbb{Z})^{13}$	$(\mathbb{Z}/3\mathbb{Z})^7$	0	-
(1,2),(1,2,2),(2,2,2)	0	0	0	0	$(\mathbb{Z}/3\mathbb{Z})^{28}$	$(\mathbb{Z}/3\mathbb{Z})^{15}$	0	-
(2,2),(1,1,2),(2,2,2)	0	0	0	0	$(\mathbb{Z}/3\mathbb{Z})^{48}$	$(\mathbb{Z}/3\mathbb{Z})^{19}$	0	-
(2,2),(1,2,2),(1,2,2)	0	0	0	0	$(\mathbb{Z}/3\mathbb{Z})^{30}$	$(\mathbb{Z}/3\mathbb{Z})^{34}$	0	-
(2,2),(1,2,3),(2,2,2)	0	0	0	0	0	$(\mathbb{Z}/3\mathbb{Z})^{59}$	$(\mathbb{Z}/3\mathbb{Z})^5$	0
(2,2),(2,2,2),(2,2,2)	0	0	0	0	0	$(\mathbb{Z}/3\mathbb{Z})^{106}$	$(\mathbb{Z}/3\mathbb{Z})^{15}$	0

And for $(2,3,3)$ over $\mathbb{Z}/5\mathbb{Z}$ we include the partitions for which we computed torsion over $\mathbb{Z}$:

Partition/Homology	0	1	2	3	4	5	6	7
(1,1),(1,1,2),(2,2,2)	0	0	0	$(\mathbb{Z}/5\mathbb{Z})^{24}$	$(\mathbb{Z}/5\mathbb{Z})^7$	0	-	-
(1,2),(1,1,1),(2,2,2)	0	0	$(\mathbb{Z}/5\mathbb{Z})^3$	$(\mathbb{Z}/5\mathbb{Z})^{39}$	$(\mathbb{Z}/5\mathbb{Z})^8$	0	-	-
(1,2),(1,1,3),(2,2,2)	0	0	0	0	$(\mathbb{Z}/5\mathbb{Z})^{12}$	$(\mathbb{Z}/5\mathbb{Z})^6$	0	-
(1,2),(1,2,2),(2,2,2)	0	0	0	0	$(\mathbb{Z}/5\mathbb{Z})^{26}$	$(\mathbb{Z}/5\mathbb{Z})^{13}$	0	-
(2,2),(1,1,2),(2,2,2)	0	0	0	0	$(\mathbb{Z}/5\mathbb{Z})^{46}$	$(\mathbb{Z}/5\mathbb{Z})^{17}$	0	-
(2,2),(1,2,2),(1,2,2)	0	0	0	0	$(\mathbb{Z}/5\mathbb{Z})^{25}$	$(\mathbb{Z}/5\mathbb{Z})^{29}$	0	-
(2,2),(1,2,3),(2,2,2)	0	0	0	0	0	$(\mathbb{Z}/5\mathbb{Z})^{58}$	$(\mathbb{Z}/5\mathbb{Z})^4$	0
(2,2),(2,2,2),(2,2,2)	0	0	0	0	0	$(\mathbb{Z}/5\mathbb{Z})^{101}$	$(\mathbb{Z}/5\mathbb{Z})^{10}$	0

Here we include an assortment of Betti numbers (over $\mathbb{Z}$, though identical for all $\mathbb{Z}/p\mathbb{Z}$ when $p \neq 3$) for the $(2,3,3)$ tripartite graph (The 1's in the 0^{th} homology group are not reflected in the homology table as the homology table's are reduced homology):

Partition/Degree	0	1	2	3	4	5	6	7	8	9	10
(1,1),(1,1,1),(1,1,1)	1	0	50	0	-	-	-	-	-	-	-
(1,1),(1,1,1),(1,1,2)	1	0	23	1	-	-	-	-	-	-	-
(1,1),(1,1,1),(1,1,3)	1	0	12	6	0	-	-	-	-	-	-
(1,1),(1,1,1),(1,2,2)	1	0	12	18	0	-	-	-	-	-	-
(1,1),(1,1,1),(1,2,3)	1	0	12	24	0	-	-	-	-	-	-
(1,1),(1,1,1),(1,3,3)	1	0	12	30	0	-	-	-	-	-	-
(1,1),(1,1,1),(2,2,2)	1	0	12	46	0	-	-	-	-	-	-
(1,1),(1,1,1),(2,2,3)	1	0	12	42	0	-	-	-	-	-	-
(1,1),(1,1,1),(2,3,3)	1	0	12	43	5	-	-	-	-	-	-
(1,1),(1,1,1),(3,3,3)	1	0	12	48	14	-	-	-	-	-	-
(1,1),(1,1,2),(1,1,2)	1	0	0	35	0	-	-	-	-	-	-
(1,1),(1,1,3),(1,1,3)	1	0	0	0	2	0	-	-	-	-	-
(1,1),(1,2,2),(1,2,2)	1	0	0	0	20	0	-	-	-	-	-
(1,1),(2,2,2),(2,2,2)	1	0	0	0	0	27	5	0	-	-	-
(1,1),(2,3,3),(3,3,3)	1	0	0	0	0	0	0	2	0	-	-
(1,1),(3,3,3),(3,3,3)	1	0	0	0	0	0	0	0	0	0	-
(1,2),(1,1,1),(1,1,1)	1	0	24	1	-	-	-	-	-	-	-
(1,2),(2,2,2),(2,2,2)	1	0	0	0	0	121	0	-	-	-	-
(1,2),(3,3,3),(3,3,3)	1	0	0	0	0	0	0	0	6	0	-
(2,2),(1,1,1),(1,1,1)	1	0	6	46	0	-	-	-	-	-	-
(2,2),(2,2,2),(2,2,2)	1	0	0	0	0	101	10	0	-	-	-
(2,2),(3,3,3),(3,3,3)	1	0	0	0	0	0	0	0	36	0	0

For the $(3,3,3)$ case we include the partitions for which we computed torsion, since we only computed partial data for this case, it is possible there is more tosion in other partitions. We leave the completion of this calculation to future work:

Partition/Homology	0	1	2	3	4	5	6	7
(1,1,1),(1,1,1),(1,1,1)	0	0	$\mathbb{Z}^{47} \bigoplus (\mathbb{Z}/3\mathbb{Z})^2$	$\mathbb{Z}^3$	-	-	-	-
(1,1,1),(1,1,1),(2,2,2)	0	0	$\mathbb{Z}^6$	$\mathbb{Z}^{81} \bigoplus (\mathbb{Z}/3\mathbb{Z})^4 \bigoplus \mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}^{20}$	0	-	-

To compare to the matching complex, here are some homologies for matching complexes of 3 partite graphs.

3-partite Matching Complex Homology

Partition/Homology	0	1	2	3	4
$K_{1,1,1}$	$\mathbb{Z}^2$	-	-	-	-
$K_{1,1,2}$	$\mathbb{Z}^2$	0	-	-	-
$K_{1,1,3}$	$\mathbb{Z}$	$\mathbb{Z}$	-	-	-
$K_{1,1,4}$	$\mathbb{Z}$	$\mathbb{Z}^5$	-	-	-
$K_{1,2,2}$	0	$\mathbb{Z}^3$	-	-	-
$K_{1,2,3}$	0	$\mathbb{Z}^8$	0	-	-
$K_{1,2,4}$	0	$\mathbb{Z}^8$	$\mathbb{Z}$	-	-
$K_{1,3,3}$	0	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}^2$	-	-
$K_{1,3,4}$	0	0	$\mathbb{Z}^{30}$	0	-
$K_{1,4,4}$	0	0	$\mathbb{Z}^{25}$	$\mathbb{Z}^2$	-
$K_{2,2,2}$	0	$\mathbb{Z}^{11}$	0	-	-
$K_{2,2,3}$	0	$\mathbb{Z}^2 \oplus \mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}^3$	-	-
$K_{2,2,4}$	0	$\mathbb{Z}^2$	$\mathbb{Z}^{35}$	0	-
$K_{2,3,3}$	0	0	$\mathbb{Z}^{50}$	0	-
$K_{2,3,4}$	0	0	$\mathbb{Z}^{45} \oplus \mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}^2$	-
$K_{2,4,4}$	0	0	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}^{217}$	0
$K_{3,3,3}$	0	0	$\mathbb{Z}^{47} \oplus \mathbb{Z}/3\mathbb{Z}^2$	$\mathbb{Z}^3$	-
$K_{3,3,4}$	0	0	$\mathbb{Z}^6 \oplus \mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}^{262}$	0
$K_{3,4,4}$	0	0	0	$\mathbb{Z}^{563} \oplus \mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}^2$

12. Triangle Matchings of Tripartite Graphs

We define the following class of simplicial complexes and hope to relate them to modules through the techniques described in section 4.

Definition 12.1. Let G be a graph, a *triangle* is a set of vertices $\{x_1, x_2, x_3\}$ such that there is an edge between any two of the vertices.

Definition 12.2. Let K be a set of triangles, we say the *degree of vertex x in K* is equal to the number of triangles in K containing x. $d_K(x) = \{T \in K | x \in T\}$

For (μ_1, μ_2, μ_3) where the length of μ_1,μ_2, and μ_3 are j, k, l respectively, we consider the tripartite graph whose components C_1, C_2, C_3 have j, k, and l vertices's respectively. Then for each component we give the vertices in them some order. Then we can associate the j^{th} vertex of C_i with the j^{th} value of μ_i. We call this the *bound of vertex* x and will denote it as $bd_{(\mu_1,\mu_2,\mu_3)}(x)$. Furthermore we will call (μ_1, μ_2, μ_3) the *multidegree* or bounded degree.

Definition 12.3. Let $T_{(\mu_1,\mu_2,\mu_3)}$ the complex whose elements K are sets of triangles such that $d_k(x) \leq bd_{(\mu_1,\mu_2,\mu_3)}(x)$.

We are bounding the number of times we allow a vertex to appear in a set of triangles, which is closed under removal of triangles from K, so the simplicial complex is well defined. When the multidegree (μ_1, μ_2, μ_3) has values all 1, then we use the notation $T_{r,s,t}$ where r,s,t are the length of μ_1, μ_2, μ_3 respectively.

Below is the data for when μ_1,μ_2, and μ_3 have entries that are all 1. The calculations are done in Sage by creating a graph whose vertices are indexed by triangles of the tripartite graph. There is an edge between two verteces if the triangles do not share a vertex. Then the desired complex is the clique complex of this graph.

3-partite Triangle Matching Complex Homology

Partition/Homology	0	1	2	3
$T_{2,2,2}$	$\mathbb{Z}^3$	0	-	-
$T_{3,2,2}$	$\mathbb{Z}$	$\mathbb{Z}^2$	-	-
$T_{3,3,2}$	0	$\mathbb{Z}^{19}$	-	-
$T_{3,3,3}$	0	$\mathbb{Z}^{46}$	0	-
$T_{4,2,2}$	$\mathbb{Z}$	$\mathbb{Z}^{10}$	-	-
$T_{4,3,2}$	0	$\mathbb{Z}^{49}$	-	-
$T_{4,3,3}$	0	$\mathbb{Z}^{43}$	$\mathbb{Z}^6$	-
$T_{4,4,2}$	0	$\mathbb{Z}^{113}$	-	-
$T_{4,4,3}$	0	$\mathbb{Z}^4$	$\mathbb{Z}^{195}$	-
$T_{4,4,4}$	0	0	$\mathbb{Z}^{927}$	0
$T_{5,2,2}$	$\mathbb{Z}$	$\mathbb{Z}^{22}$	-	-
$T_{5,3,2}$	0	$\mathbb{Z}^{91}$	-	-
$T_{5,3,3}$	0	$\mathbb{Z}^{40}$	$\mathbb{Z}^{84}$	-
$T_{5,4,2}$	0	$\mathbb{Z}^{201}$	-	-
$T_{5,4,3}$	0	$\mathbb{Z}^2$	$\mathbb{Z}^{781}$	-
$T_{5,4,4}$	0	0	$\mathbb{Z}^{1543}$	$\mathbb{Z}^{24}$
$T_{5,5,2}$	0	$\mathbb{Z}^{351}$	-	-

We also have code and data for other values of the multidegree μ_1,μ_2, and μ_3 are greater then 1 but we will omit it because of the amount.

Theorem 12.4. *Given complex $T_{(\mu_1,\mu_2,\mu_3)}$ where $\mu_i \vdash n_i$, all maximal simplices will have dimension $min\{n_1, n_2, n_3\} - 1$*

Proof. Given a element L of the $T_{(\mu_1,\mu_2,\mu_3)}$ with less then $min\{n_1, n_2, n_3\}$ triangles, there will exist a vertex x_1, x_2, x_3 in C_1, C_2, C_3 respectively such that $d_L(x_i) < bd_{(\mu_1,\mu_2,\mu_3)}(x_i)$ because of our choice of bound. Then $L \cup \{x_1, x_2, x_3\}$ is also in the complex. If a set of triangles has order grater then $min\{n_1, n_2, n_3\}$ then $d_L(x_i) > bd_{(\mu_1,\mu_2,\mu_3)}(x_i)$ for some x_i by the pigeonhole principal and so is not in the complex. □

Theorem 12.5. *Let multidegree μ_1, μ_2, μ_3 of length n_1, n_2, n_3 have entries all 1. Then $T_{(\mu_1,\mu_2,\mu_3)}$ has $\binom{n_1}{k} \cdot \frac{n_2!}{(n_2-k)!} \cdot \frac{n_3!}{(n_3-k)!}$ faces of dimension $k-1$ for $0 \le k \le min\{n_1, n_2, n_3\} - 1$.*

Proof. We can make a set of k triangles on the complete tripartite graph K_{n_1,n_2,n_3} that do not overlap uniquely by the following method. Pick k vertices from each of the components $x_1, x_2, ..., x_k \in C_1$, $y_1, y_2, ..., y_k \in C_2$, and $z_1, z_2, ..., z_k \in C_3$. We can make a unique set of k triangles by the following. Fix an arbitrary order for the vertices from C_1. Without loss of generality lets order them as $x_1, ..., x_k$. Then for each ordering of vertices $y_1, y_2, ..., y_k$ and $z_1, z_2, ..., z_k$ as $y_{i_1}, ..., y_{i_k}$ and $z_{j_1}, ..., z_{j_k}$, take the set of triangles $\{\{x_1, y_{i_1}, z_{j_1}\}, \{x_2, y_{i_2}, z_{j_2}\}, ..., \{x_k, y_{i_k}, z_{j_k}\}\}$. This will uniquely define a k-1 dimensional simplex since we fixed the order of vertices from C_1. Now we can count the number of orderings, there are $\binom{n_1}{k}\binom{n_2}{k}\binom{n_3}{k}$ initial choices and once we fix an ordering on C_1 the there are $k!$ choices of ordering for $y_1, y_2, ..., y_k$ and $z_1, z_2, ..., z_k$ each. So then we have $\binom{n_1}{k}\binom{n_2}{k}\binom{n_3}{k} \cdot k! \cdot k! = \binom{n_1}{k} \cdot \frac{n_2!}{(n_2-k)!} \cdot \frac{n_3!}{(n_3-k)!}$ □

We now look at some results in the table above. We conjecture from the results that for the case of μ_1, μ_2, μ_3 all of length n have entries all 1, and that the reduced homology is trivial everywhere except at degree $n-2$. Then from the following formula over a field

$$\sum(-1)^k \dim(C_i) = \sum(-1)^k \dim(H_i(X))$$

where C_i is the i^{th} dimension of the simplicial complex and along with the conjecture that the homology will be torsion free we get the following:

Conjecture 12.6. *Let multidegree μ_1, μ_2, μ_3 all be of length n have entries all 1, then T_{μ_1,μ_2,μ_3} has the homology as follows*

$$H_i(T_{\mu_1,\mu_2,\mu_3}) = \begin{cases} 0 & i \neq n-2 \\ \mathbb{Z}^{a(n)} & i = n-2 \end{cases}$$

where $a(n) = (-1)^{n-2} \sum_{k=0}^{n} (-1)^k \binom{n}{k} (\frac{n!}{(n-k)!})^2$.

Now we look at the case when $\mu_3 = (1,1)$ and let the μ_1, μ_2 be length $n_1 \geq 2$ and $n_2 \geq 2$ and have all one entries. Then the maximal dimension of the simplicial complex is 1. Construct the graph whose vertex set is the triangles of $K_{n_1,n_2,2}$ and an edge between triangles if they do not share a vertex. Then the simplicial complex of this graph is T_{μ_1,μ_2,μ_3}. Then we the following are known about the complex.

Theorem 12.7. *Let $\mu_3 = (1,1)$, μ_1, μ_2 be length $n_1 \geq n_2 \geq 3$ and have all 1 entries. Then (reduced homology) $H_0(T_{\mu_1,\mu_2,\mu_3}) = 0$. If $n_1 \geq 3$ and $n_2 = 2$ then we have that $H_0(T_{\mu_1,\mu_2,\mu_3}) = \mathbb{Z}$. Lastly if $n_1 = n_2 = 2$ the $H_0(T_{\mu_1,\mu_2,\mu_3}) = \mathbb{Z}^3$*

Proof. We will prove the number of connected components the graph has, which will give us the homology H_0. When $n_1 \geq n_2 \geq 3$ let the vertices of the tripartite graph be $x_1, ..., x_{n_1} \in C_1$, $y_1, ..., y_{n_2} \in C_2$ and $z_1, z_2 \in C_3$. We will prove all vertices are connected to the vertex indexed by the triangle $\{x_1, y_1, z_1\}$. Given a $\{x_i, y_j, z_k\}$ if $i, j, k \neq 1$ then there will be an edge between the vertex it indexes and the vertex indexed by $\{x_1, y_1, z_1\}$. If $k = 1$ then there exists a $i' \neq i \neq 1$ and $j' \neq j \neq 1$ so then $\{x_i, y_j, z_1\}$ has an edge to $\{x_{i'}, y_{j'}, z_2\}$ which we have already said has an edge to $\{x_1, y_1, z_1\}$. Then for triangles of form $\{x_i, y_j, z_2\}$ where i or j is 1, then it is connected to some $\{x_{i'}, y_{j'}, z_1\}$ which is connected to $\{x_1, y_1, z_1\}$. There is only one connected component of the graph so the reduced homology is 0.
Then when $\mu_2 = (1,1)$ we have that $x_1, ..., x_{n_1} \in C_1$, $y_1, y_2 \in C_2$ and $z_1, z_2 \in C_3$. Then it can be seen that there is no sequence of edges connecting $\{x_1, y_1, z_1\}$ and $\{x_1, y_1, z_2\}$ since any triangle indexing a vertex connected by an edge will simultaneously switch the vertices's of C_2 and C_3. Then it can be seen by similar argument to $n_2 \geq 3$ case that every vertex will be connected to one of $\{x_1, y_1, z_1\}$ or $\{x_1, y_1, z_2\}$. There are two connected components of the graph so the reduced homology is $\mathbb{Z}$.
For $n_1 = n_2 = 2$ one can draw the easily graph and see that there are 8 vertices's and 4 disjoint components giving us the desired homology result. □

Theorem 12.8. *Let $\mu_3 = (1,1)$, μ_1, μ_2 be length $n_1 \geq 3$ and $n_2 \geq 3$ and have all 1 entries. Then $H_1(T_{\mu_1,\mu_2,\mu_3}) = \mathbb{Z}^{b(n_1,n_2)}$, where $b(n_1, n_2) = 1 - 2n_1n_2 + n_1(n_1-1)n_2(n_2-1)$. If $n_1 \geq 3$ and $n_2 = 2$ then $H_1(T_{\mu_1,\mu_2,\mu_3}) = \mathbb{Z}^{b(n_1,n_2)+1}$. When $n_1 = n_2 = 2$ then $H_1(T_{\mu_1,\mu_2,\mu_3})$ is trivial.*

Proof. Since we are just looking at the complex of a graph it suffices to not find cycles to show if the homology is trivial. For the $n_1 = n_2 = 2$ case draw the graph as in the proof of theorem 12.7. The graph will have no loops so the kernel of the boundary map will be 0.
Then recall for complexes of graphs for each connected component the the 1st homology will be a free module rank $|E| - |V| + 1$. Furthermore we can take direct sums in the case of disjoint components. The using Theorem 12.5 and plugging in $n_3 = 2$ we get $1 - 2n_1n_2 + n_1(n_1-1)n_2(n_2-1)$ if we have one component. We will not have one component in the case of $n_1 \geq 3$ and $n_2 = 2$ as explained in the proof of theorem 12.7. In this case we have two components with $|E| - |V| + 1$ values $1 - n_1n_2 + \frac{1}{2}n_1(n_1-1)n_2(n_2-1)$ in both since the two components will be identical. Then adding the two will give us $b(n_1, n_2) + 1$. □

To relate this back to Section 4, we consider the subalgebra of $k[x_1, ..., x_i, y_1, ..., y_j, z_1, ..., z_k]$ generated by monomials of form $x_r y_s z_t$ where $1 \le r \le i$, $1 \le s \le j$, $1 \le t \le k$. Lets call this subalgebra $Triple(i, j, k)$ for now. Using the notation from Section 4, we let the module be $Triple(i, j, k)$ and we consider the special case when $i = j = k$. Then the module defined in Section 4, K_{μ_1,μ_2,μ_3} is non-empty when μ_1, μ_2, μ_3 have entries all one, so they corresponds to a weight space of $Tor_i^A(Triple(n, n, n), k)$ for $A = k[x_1, ..., x_n, y_1, ..., y_n, z_1, ..., z_n]$. K_{μ_1,μ_2,μ_3} will be isomorphic to T_{μ_1,μ_2,μ_3} in this case. We are looking into cases when i, j, k are not the same value and the the case when the multidegrees in T_{μ_1,μ_2,μ_3} contain values greater than 1.

13. Code and Documentation

lessthanvector(u,v)

- u,v - vectors inputs to be compared. They should be the same length, code will still work if u has more components then v.

Compares two vectors, returns True if the ith component of u is less then of equal to the ith component of v for all components in u.

allonevector(i)

- i - positive integer

Returns a vector length i with all entry as 1.

Boundedgraphelement(Vector,Graph)

- Vector-vector, input should be the bounded degree of each vertex of Graph in the order they are given in Graph.
- Graph - Any non-empty simple graph without a loop, vertex ordering should correspond to bound given in vector.

Returns True if the Graph inputed meets the vertex bound given by vector. Returns False otherwise.

Boundedgraphelement(Vector,Graph)

- Vector-vector, input should be the bounded degree of each vertex of Graph in the order they are given in Graph.
- Graph - Any non-empty simple graph with a loop, vertex ordering should correspond to bound given in vector.

Returns True if the Graph inputed meets the vertex bound given by vector. Returns False otherwise. This is the code with adjustments to account for the 2 degrees a loop adds to the degree of a vertex. We replace Boundedgraphelement(Vector,Graph) with this function in AddGraphWithEdgeToList(List,edge,boundvector) to adjust the other code to loops.

AddGraphWithEdgeToList(List,edge,boundvector)

- List - Initial list of simple graphs without loops
- edge - edge to be added and checked if resulting graph meets bound requirements. The edge must be have endpoints in verteces already present in graphs in List.
- boundvector - vector, maximum vertex degree condition to check.

Returns a list of graphs which includes thee original List and any graph in List with edge added that still meet the bounded degree condition.

GenerateAllBoundedDegreeGraphs(OriginalGraph,boundeddegree)

- OriginalGraph - Simple Graph(without loops) that we wish to find all embeddable subgraphs which meet a maximum vertex degree condition

- boundeddegree - vector input indexed by verteces of OriginalGraph indicating the maximum degree we allow subgraphs of OriginalGraph to have.

Returns a list of subgraphs of OriginalGraph that meet the maximum degree bound given by boundeddegree.

```
#Checks number of components in u<v, then check if that number is equal to the length of
    u.
def lessthanvector(u,v):
    i = len(u)
    counter=0
    k=0
    while k<i:
        if u[k] < v[k]+1:
            counter = counter+1
        k=k+1
    return counter==i

#make all 1 vector length i
def allonevector(i):
    return matrix(QQ, 1, i, lambda x, y: 1).row(0)

#Then we construct a function to check if a graph is bounded as such we construct one
    for the adjacency matrix A for a graph G by degree v(a partition)
def Boundedgraphelement(vector,Graph):
    if len(vector) == len(Graph):
        A = Graph.adjacency_matrix()
        u = A*allonevector(len(vector))
        return lessthanvector(u,vector)
    else:
        return False


def Boundedgraphelementwithloop(vector,Graph):
    if len(vector) == len(Graph):
        g = copy(Graph)
        A = g.adjacency_matrix()
        g.remove_loops()
        if g.remove_loops == None:
            u = 2*A
        else:
            B = A-g.adjacency_matrix()
            u = (A+B)*allonevector(len(vector))
        return lessthanvector(u,vector)
    else:
        return False


def AddGraphsWithEdgeToList(List,edge,boundvector):
    DummyList = []
    for Graph in List:
        AddEdgeGraph = copy(Graph)
        AddEdgeGraph.add_edge(edge)
        if Boundedgraphelement(boundvector,AddEdgeGraph):
            DummyList.append(AddEdgeGraph)
```

```
    List.extend(DummyList)

def GenerateAllBoundedDegreeGraphs(OriginalGraph,boundeddegree):
    InitialGraph = Graph(OriginalGraph.order())
    List = [InitialGraph]
    for edge in OriginalGraph.edges():
        AddGraphsWithEdgeToList(List,edge,boundeddegree)
    return List
```

---

checkifsimplex(Y)

- Y - the object to be tested, preferably a set or simplicial complex.

Returns True if input is a simplicial complex and is a simplex at least on 0-dimensional element or empty. Returns False otherwise, most importantly in the case when the simplicial complex is not a simplex, when it is the simplicial complex containing only the empty set and when the input is a set.

IsnonEvasive(Set,Order,Complex)

- Set - The simplex expressed as a set of 0-simplices, to be assessed as evasive or non-evasive.
- Order - A some permutation of the 0-simpilices that will determine a order of decomposition for the element decision tree.
- Complex - Simplicial Complex we are making a decision tree for.

Returns True if the Set if Set is non-evasive in the decision tree of Complex determined by Order. The order gives the order of decomposition into link and deletion of the complex.

numberofEvasive(Order,Complex)

- Order - Some permutation of the 0-simplices which determine the order of decompostionfor a decision tree for Complex
- Complex - Complex to be assessed

Returns the number of evasive sets in Complex given the decision tree for Complex determined by Order.

numberofEvasivewithdim(Order,Complex,dim)

- Order - Some permutation of the 0-simplices which determine the order of decompostionfor a decision tree for Complex
- Complex - Complex to be assessed
- dim - dimension of Complex to find number of evasive sets in

Returns the number of evasive sets in dimension dim of Complex given the decision tree for Complex determined by Order.

MinimalEvasiveOverRandomPermutation(Complex)

- Complex - A finite simplicial complex

Returns the minimal number of evasive sets in Complex given the decision tree for Complex determined by Order, where we run through all permutations for Order. This does not give us the true minimum over all decision trees but will give us a bound on the minimal number of evasive sets.

MinimalEvasiveOverRandomPermutationwithdim(Complex,dim)

- Complex - A finite simplicial complex
- dim - integer indicating a dimension of Complex.

Returns the minimal number of evasive sets in dimension dim of Complex given the decision tree for Complex determined by Order, where we run through all permutations for Order. This does not give us the true minimum over all decision trees but will give us a bound on the minimal number of evasive sets of dimension dim.

NumEvasiveOverRandomPermutation(Complex)

- Complex - Complex to be assessed

Returns the number of evasive sets in Complex given the decision tree for Complex determined by Order, which is a random permutation of the 0-simplices.

NumEvasiveOverRandomPermutationwithdim(Complex,dim)

- Complex - Complex to be assessed
- dim - integer indicating dimension of elements of Complex

Returns the number of evasive sets in Complex of dimension dim given the decision tree for Complex determined by Order, which is a random permutation of the 0-simplices.

```
def checkifsimplex(Y):
    if isinstance(Y,SimplicialComplex):
        if Y == SimplicialComplex():
            return False
        elif len(Y.maximal_faces())==1:
            return True
        else: return False
    elif Y.is_empty():
        return True
    else:
        return False


#determine if a set is nonevasive input
def IsnonEvasive(Set,Order,Complex):
    #print(Complex.faces())
    NewComplex = copy(Complex)
    NewOrder = copy(Order)
    NewSet = copy(Set)
    if len(NewOrder) == 0:
        #print("A")
        return False
    elif NewOrder[0] in Set:
        #NewComplex = Complex
        NewComplex = NewComplex.link([NewOrder[0]])
        if checkifsimplex(NewComplex) == True:
            #print("B")
            return True
        else:
            #print("C")
            NewSet.remove(NewOrder[0])
            NewOrder.remove(NewOrder[0])
            return IsnonEvasive(NewSet,NewOrder,NewComplex)
    else:
        NewComplex.remove_face([NewOrder[0]])
        if checkifsimplex(NewComplex) == True:
```

```
            #print("E")
            return True
        else:
            #print("F")
            NewOrder.remove(NewOrder[0])
            return IsnonEvasive(NewSet,NewOrder,NewComplex)


def numberofevasive(Order,Complex):
    count = 0
    i = len(Complex.faces())-1
    for j in range(i):
        #print Complex.faces()[j]
        for Set in Complex.faces()[j]:
            #print(Order)
            #print(IsnonEvasive(list(Set),Order,Complex))
            if IsnonEvasive(list(Set),Order,Complex) == False:
                count = count+1
            #print("A")
            #print(Set)
            #print(count)
    return count

def numberofevasivewithdim(Order,Complex,Dimension):
    count = 0
    for Set in Complex.faces()[Dimension]:
            #print("B")
            if IsnonEvasive(list(Set),Order,Complex) == False:
                count = count+1
                #print("A")
            #print(count)
    return count

def MinimalEvasiveOverPermutations(Complex):
    k = -1
    for Order in Permutations(len(Complex.faces()[0])):
        Order = list(Order)
        l = numberofevasive(Order,Complex)
        if k == -1:
            k=l
        elif k>l:
            k=l
    return k

def MinimalEvasiveOverPermutationswithdim(Complex,Dimension):
    k = -1
    for Order in Permutations(len(Complex.faces()[0])):
        Order = list(Order)
        l = numberofevasivewithdim(Order,Complex,Dimension)
        if k == -1:
            k=l
        elif k>l:
            k=l
    return k
```

```
def NumEvasiveOverRandomPermutation(Complex):
    Order = Permutations(len(Complex.faces()[0])).random_element()
    print(Order)
    Order = list(Order)
    return numberofevasive(Order,Complex)

def NumEvasiveOverRandomPermutationwithdim(Complex,Dimension):
    Order = Permutations(len(Complex.faces()[0])).random_element()
    print(Order)
    Order = list(Order)
    return numberofevasivewithdim(Order,Complex,Dimension)
```

---

matchinghom(g)

- g - Graph

Returns the homology of the simplicial complex whose elements are partial matchings of the graph g.

```
def matchinghom(g):

    h = g.line_graph()
    G = h.complement()
    X = G.clique_complex()

    return X.homology()
```

---

generatealltriangles(r,s,t)

- r,s,t - integer entries counting the number of vertex in the tripartite graph

Returns the number of triangles in a tripartite graph with r,s,t vertices's in its components. Triangles are triples such that any two elements of the triple are connected by an edge to each other.

sharevertex(list1,list2)

- list1, list2 - list of integers

Returns True if the i^{th} components of list1 is the same as the i^{th} component of list2 for all index of list1.

maketrianglerelationgraph(List)

- List - List of lists of integers, ideally this is a list of triangles(as a list of vertices's)

Returns a graph whose vertex are indexed by the items of List. There is an edge between two vertices's if they share a common element in the i^{th} component. In terms of triangles on a tripartite graph, the vertices's of triangles and there is an edge if they share a vertex.

maketrianglematchingcomplex(r,s,t)

- r,s,t - positive integers

Returns the simplicial complex whose elements are sets of triangles of a who do not share any vertices's.

checkboundcondition(matchingbound1,matchingbound2,matchingbound3,bound1,bound2,bound3,t

- bound1, bound2, bound3 - list of positive integers, indicates the vertex bounds of each vertex in the respective component

- matchingbound1, matchingbound2, mathchingbound3 - list of list of bounds
- triangle - list of 3 positive integers indexing the triangle form the complete tripartite graph

Returns True if the given triangle meets the bound requirements when added to the bounds for another list of triangles.

addmatchingtolists(ListOfBoundedMatchings,ListofBounds1,ListofBounds2,ListofBounds3,index,tr

- ListOfBoundedMatchings - List of sets of triangles of a tripartite graph
- ListofBounds1,ListofBounds2,ListofBounds3 - corresponding degrees for the sets of triangle in ListOfBoundedMatchings
- triangle - triangle to be added
- index - index of list of triangles to have a triangle added to
- triangle - list of 3 tuple that make a triangle to be added

Adds new set of triangle to ListOfBoundedMatchings and updates List OfBounds with the corresponding degrees for the components.

generateboundeddegreetriangle(bound1,bound2,bound3)

- bound1,bound2,bound3 - list of integers giving bounds of each vertex in the triparite graph by component

Returns a list of sets of triangles where each vertex occurs at most by the bound given.

```
import itertools

def generatealltriangles(r,s,t):
    K = []
    for i in range(r):
        for j in range(s):
            for k in range(t):
                K.append([i,j,k])
    return K

def sharevertex(list1,list2):
    for i in range(len(list1)):
        if list1[i] ==list2[i]:
            return True
    return False

def maketrianglerelationgraph(List):
    K = []
    size = len(List)
    everypair = list(itertools.combinations(range(size), 2))
    for pair in everypair:
        if sharevertex(List[pair[0]],List[pair[1]])==False:
            K.append(pair)
    RelationGraph = Graph(K)
    return RelationGraph

def maketrianglematchingcomplex(r,s,t):
    Triangles = generatealltriangles(r,s,t)
    RelationGraph = maketrianglerelationgraph(Triangles)
    TriangleMatchingComplex = RelationGraph.clique_complex()
    return TriangleMatchingComplex
```

```
def
    checkboundcondition(matchingbound1,matchingbound2,matchingbound3,bound1,bound2,bound3,triangle):
    if matchingbound1[triangle[0]]<bound1[triangle[0]]:
        if matchingbound2[triangle[1]]<bound2[triangle[1]]:
            if matchingbound3[triangle[2]]<bound3[triangle[2]]:
                return True
    else:
        return False


def
    addmatchingtolists(ListOfBoundedMatchings,ListofBounds1,ListofBounds2,ListofBounds3,index,triangle,t
    bound1 = ListofBounds1[index]+vector(ZZ,
        {len(ListofBounds1[index])-1:0,triangle[0]:1})
    ListofBounds1.append(bound1)
    bound2 = ListofBounds2[index]+vector(ZZ,
        {len(ListofBounds2[index])-1:0,triangle[1]:1})
    ListofBounds2.append(bound2)
    bound3 = ListofBounds3[index]+vector(ZZ,
        {len(ListofBounds3[index])-1:0,triangle[2]:1})
    ListofBounds3.append(bound3)
    dummymatching = copy(ListOfBoundedMatchings[index])
    dummymatching.append(triangleindex)
    ListOfBoundedMatchings.append(dummymatching)

#make a function to check vertex bound conditions
def generateboundeddegreetriangles(bound1,bound2,bound3):
    r = len(bound1)
    s = len(bound2)
    t = len(bound3)
    AllTriangles = generatealltriangles(r,s,t)
    ListOfBoundedMatchings = [[]]
    ListofBounds1 = [vector(ZZ, {r-1:0})]
    ListofBounds2 = [vector(ZZ, {s-1:0})]
    ListofBounds3 = [vector(ZZ, {t-1:0})]
    for j in range(len(AllTriangles)):
        for i in range(len(ListOfBoundedMatchings)):
            if
                checkboundcondition(ListofBounds1[i],ListofBounds2[i],ListofBounds3[i],bound1,bound2,boun
                addmatchingtolists(ListOfBoundedMatchings,ListofBounds1,ListofBounds2,ListofBounds3,i,All
    return ListOfBoundedMatchings
```

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