UNIVERSITY OF Nebraska Lincoln

New Perspectives on Geproci-ness

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Understanding Complete Intersections

Definition

An algebraic set is a **complete intersection** if it equal to the intersection of algebraic sets.

For this talk, we will be interested in complete intersections that are sets of **points**.

Like this.

What is Geproci?

Definition

An algebraic set Z in $\mathbb{P}_k^n$ is **geproci** if the projection of Z from a general point P onto a hyperplane is a complete intersection in $\mathbb{P}_k^{n-1}$.

Geproci stands for **ge**neral **pro**jection is a **c**omplete **i**ntersection.

What We Know: Coplanar Points

A set of coplanar points can only be geproci if they are already a complete intersection on the plane they're on.

What We Know: Grids

Definition

A **grid** in $\mathbb{P}^3$ is a set of points that form the intersection of two families of mutually-skew lines.

Every grid is geproci, and the projection of the points of a grid is a complete intersection of two unions of lines.

Grids and coplanar points are the trivial cases of geproci-ness.

An (a, b)-grid with $3 \leq a \leq b$ is a set of points on a smooth quadric.

What We Know: D_4

D_4 is a set of 12 points and 16 3-rich lines. It is $(3, 4)$-geproci and the smallest non-trivial geproci set in characteristic 0.

D_4 is a *half-grid*. It is also the only non-trivial $(3, b)$-geproci set where $b \geq 3$.

Unexpected Cones

All known examples of geproci sets are from unexpected cones.

Definition

A set of points $Z \subseteq \mathbb{P}^3$ admits an **unexpected cone** of degree d if

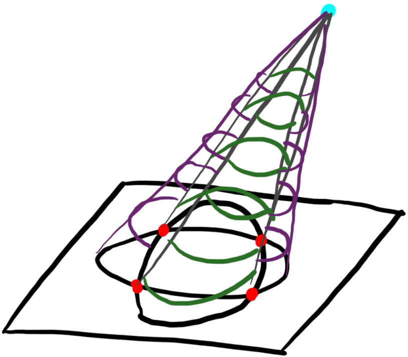
$$\dim\left([I(Z)]_d \cap [I(P)^d]_d\right) > \max\left(0, [I(Z)]_d - \binom{d+2}{3}\right)$$

for a general $P \in \mathbb{P}^3$.

This is "unexpected" because one would expect by a dimension count that being singular at P to impose $\binom{d+n-1}{n}$ conditions.

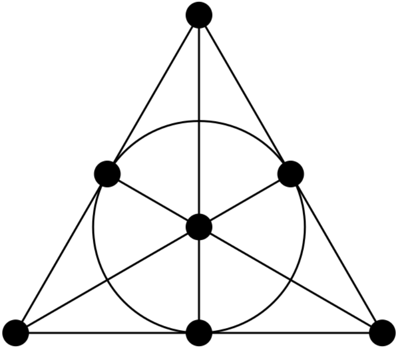
Cones and Geproci

Unexpected cones allow us to project from the general vertex onto a plane:



Geometry in Positive Characteristic

Geometry gets weird in positive characteristic! You may already be familiar with $\mathbb{P}^2_{\mathbb{Z}/2\mathbb{Z}}$, aka the **Fano Plane**.



Unexpected Cones in $\mathbb{P}^3_{\mathbb{F}_q}$

$\#\mathbb{P}^3_{\mathbb{F}_q} = \dfrac{q^4-1}{q-1} = q^3+q^2+q+1 = (q+1)(q^2+1)$. There is a degree-$q+1$ cone containing $\mathbb{P}^3_{\mathbb{F}_q}$ and has a vertex at a general point $P=(a,b,c,d)$. This cone is given by

$$\begin{aligned}(c^qd-cd^q)(x^qy-xy^q)-(b^qd-bd^q)(x^qz-xz^q)\\ +(b^qc-bc^q)(x^qw-xw^q)+(a^qd-ad^q)(y^qz-yz^q)\\ -(a^qc-ac^q)(y^qw-yw^q)+(a^qb-ab^q)(z^qw-zw^q)\end{aligned}$$

So $\dim\left([I(Z)]_{q+1}\cap[I(P)^{q+1}]_{q+1}\right) = 1 > 6-\binom{q+3}{3}$.

Spreads in $\mathbb{P}^3_{\mathbb{F}_q}$

Each line contains $q+1$ points. Can $\mathbb{P}^3_{\mathbb{F}_q}$ be partitioned by q^2+1 mutually-skew lines? Yes! Such a partition is called a **spread**.

Theorem (Bruck and Bose '63)

Let $\mathbb{P}^{2t-1}_{\mathbb{F}_q}$ be an odd-dimensional projective space over a field $\mathbb{F}_q$ of size q, where q is a power of a prime. Then there exists a spread in $\mathbb{P}^{2t-1}_{\mathbb{F}_q}$.

A Theorem

Theorem

The set of points $\mathbb{P}^3_{\mathbb{F}_q}$ is $(q+1, q^2+1)$-geproci in $\mathbb{P}^3_k$, where k is an algebraically closed field containing $\mathbb{F}_q$.

This set is a half-grid. Note when $q = 2$, we get a non-trivial $(3, 5)$-geproci set!

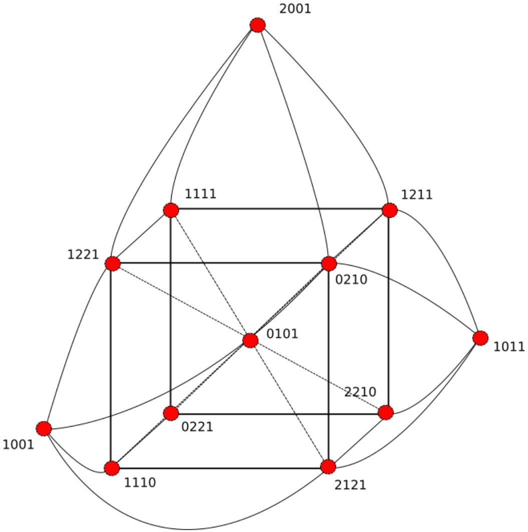
Partial Spreads

Definition

A **partial spread** of $\mathbb{P}^3_{\mathbb{F}_q}$ with deficiency d is a set of $q^2 + 1 - d$ mutually-skew lines. A **maximal partial spread** is a partial spread of positive deficiency that is not contained in a spread.

In $\mathbb{P}^3_{\mathbb{Z}/3\mathbb{Z}}$, the only maximal partial spread has seven lines ($d = 3$). The complement of this maximal partial spread is a set of 12 points that form a D_4! Recall that this configuration exists in characteristic 0.

Look



Blow-Ups of Spaces

$$\mathsf{BL}_B(\mathbb{P}^n) = \{(P, L) \in \mathbb{P}^n \times \mathsf{Gr}(2, n+1) : B \in L, P \in L\}.$$

$\mathsf{BL}_B(\mathbb{P}^n)$ projects onto $\mathbb{P}^n$ via $\pi_B(P, L) = P$.

Definition

The preimage $\pi_B^{-1}(B)$ is the **exceptional locus** of B.

For a general variety $X \hookrightarrow \mathbb{P}^n$, $\mathsf{BL}_B(X) = \overline{\pi_B^{-1}(X \setminus \{B\})}$.

Infinitely-Near Points

Definition

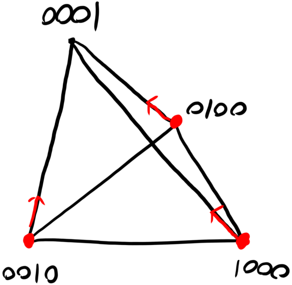
Let X be an algebraic variety and let $P \in X$. The point Q is **infinitely-near** P if Q is on the exceptional locus of the blowup of X at P.

Abuse of notation: Technically, $Q \in \mathsf{BL}_P(X)$, but we will be speaking of infinitely-near points as if they were points of X itself.

Geproci With Infinitely-Near Points

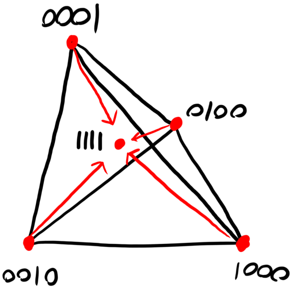
The char $k = 2$. Let $Z = \{(1,0,0,0) \times 2, (0,1,0,0) \times 2, (0,0,1,0) \times 2\}$, with the infinitely-near point at each ordinary point corresponding to the line containing $(0,0,0,1)$.

Then Z is a $(2,3)$-geproci half-grid.



Another Example

Let $Z = \{(1,0,0,0) \times 2, (0,1,0,0) \times 2, (0,0,1,0) \times 2, (0,0,0,1) \times 2, (1,1,1,1)\}$, which each infinitely-near point corresponding to the line containing $(1,1,1,1)$. Then Z is a $(3,3)$-geproci. It is not a half-grid.



Future Problems

1. Do infinitely-near points provide new examples of geproci sets in characteristic 0?
2. Does taking higher-order infinitely-near points provide new examples of geproci sets?
3. Do **maximal partial spreads** provide new examples of geproci sets that work in characteristic 0?