

A BIOGRAPHY OF THE MARKUS-YAMABE CONJECTURE¹

[1960–1995]

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Abstract

A recent new attempt by Deng, Meisters & Zampieri [29,34] to prove Keller’s 1939 **Jacobian Conjecture** (JC), although so far falling short of that goal, has nevertheless led to the discovery of surprisingly simple and unexpected *polynomial counterexamples* in \mathbb{R}^3 to the 1960 **Markus-Yamabe Conjecture** (MYC) on global asymptotic stability. These *polynomial examples* were found by van den Essen and Hubbers in Nijmegen, with help from Cima, Gasull, and Mañosas in Barcelona, using Deng’s 1995 rediscovery of Rosay & Rudin’s 1988 criterion for global analytic linearization of analytic maps. The MYC had already been proved in \mathbb{R}^2 : First in 1987 for *polynomial* vector fields by Meisters & Olech; then in 1993 for \mathcal{C}^1 vector fields—independently by Feßler, Gutierrez, and Glutsyuk. These serendipitous polynomial counterexamples to MYC in \mathbb{R}^3 are the final pieces to that 35-year-old puzzle. In spite of many “reductions”, Keller’s JC remains open.

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1. Introduction to the Markus-Yamabe Conjecture

During the academic year 1960–61, I held a Research Fellowship at RIAS² in Baltimore, Maryland, where, in the Fall of 1960, Larry Markus gave a talk on a paper [1] he had recently coauthored with Hidehiko Yamabe [2,3]. In their paper [1], Markus & Yamabe clearly stated the following conjecture, which they attributed to Aizerman [4], and which they proved under some rather strong additional conditions. This is now widely known as **The Markus-Yamabe Conjecture on global asymptotic stability**, or just MYC for short. An $n \times n$ matrix is called **stable** if all its eigenvalues have negative real parts. An old and elementary result says x_0 is a *local attractor* of $\dot{x} = F(x)$ if $F(x_0) = 0$ and the Jacobian matrix $F'(x_0)$ is stable.

MYC. *If a C^1 map $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ satisfies $f(0) = 0$ and if its Jacobian matrix $f'(x)$ is **stable** $\forall x \in \mathbb{R}^n$, then 0 is a **global attractor** of the system $\frac{dx}{dt} = f(x)$. A system satisfying these hypotheses is called an MY-SYSTEM.*

The most important results obtained on this problem in those times were those of Czesław Olech [5] for the *two-dimensional case*: (1) If, also, the vector field f is bounded away from zero in a neighborhood of infinity, then 0 is a global attractor; and (2) The desired conclusion follows if and only if the mapping $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is globally one-to-one. Other papers inspired that year by MYC included, Hartman [6], Hartman & Olech [7], and Meisters & Olech [8]. In 1978 I learned of Keller's (older) Jacobian Conjecture [9]

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and wrote [10] comparing it to MYC. This stirred up new interest, but in spite of these and many other papers [11] written in the years that followed, no definitive result was obtained until 1987 when we proved MYC for all 2-dimensional *polynomial* vector fields [12]. In 1988 Barabanov [13] gave ideas for constructing a class \mathcal{C}^1 MY-SYSTEM in 4 dimensions with a nonconstant *periodic* orbit,—and hence a counterexample to MYC in \mathbb{R}^4 . But the details of his paper were in some doubt; anyway, still open was the \mathcal{C}^1 case in 2- & 3-dimensions and the polynomial case in dimensions ≥ 3 . All of these questions were finally resolved by the end of 1995; the final piece to the MYC-puzzle was only found (as frequently happens, serendipitously!) by a discovery made while in pursuit of the solution to a different problem open since 1939: Namely, **Keller’s Jacobian Conjecture** (JC). So we must temporarily interrupt our story of the Markus-Yamabe Conjecture in order to fill in the necessary background for its serendipitous conclusion; we shall return to MYC in §3 and §5.

2. Keller’s Jacobian Conjecture

Call a polynomial map $F : \mathbb{C}^n \rightarrow \mathbb{C}^n$ a **Keller map** if $\det F'(x) \equiv 1$. If $\det F'(x) \neq 0$ on \mathbb{C}^n , then $\det F'(x)$ is a nonzero constant; so we may assume $\det F'(x) \equiv 1$. On \mathbb{R}^n these are two different conditions (see §7).

In 1939 Ott-Heinrich Keller [1906–1990] posed the following question [9].

JC. *Is every Keller map automatically bijective with polynomial inverse?*

EXAMPLE 1. The hypothesis *polynomial* cannot be replaced by *analytic*.

Define $F : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ by $F(x, y) = (u(x, y), v(x, y))$ and

$$F : \begin{cases} u(x, y) = \sqrt{2} \exp(x/2) \cos(y \exp(-x)), \\ v(x, y) = \sqrt{2} \exp(x/2) \sin(y \exp(-x)). \end{cases} \quad (1)$$

Then $F(0, y+2\pi) = F(0, y)$ so F is not injective even though $\det F'(x) \equiv 1$.

Note. *The converse of JC is true and easy to prove.*

Proof. Suppose G is a polynomial inverse of F . Then, for all $x \in \mathbb{C}^n$,

$$I = x' = [G(F(x))]' = G'(F(x))F'(x), \quad \text{so } 1 = \det G'(F(x)) \det F'(x).$$

The polynomial $\det F'(x)$ has no complex root, so it's a nonzero constant!

2.1 Reductions

The first reduction came in 1962 from Białynicki-Birula & Rosenlicht [14]; and Rudin [15] gave it a more elementary proof in 1995.

Reduction 1. *Injective polynomial maps are automatically surjective; and, at least in the complex case, their inverses are polynomial maps. Thus to prove JC it suffices to prove that a Keller map F is injective.*

Call a bijective polynomial map, whose inverse is also a polynomial map, a

polyomorphism;—short for **polynomial automorphism**.

EXAMPLE 2. Triangular polynomial maps T are polyomorphisms.

$$T : \begin{cases} u(x, y, z) = x + f(y, z), \\ v(x, y, z) = y + g(z), \\ w(x, y, z) = z. \end{cases} \quad (2)$$

Here we assume that f and g are polynomials.

Exercise. *Invert this map T . Is T^{-1} also upper triangular? Compute the Jacobian matrix T' and $\det T'$. Note that the composition of upper and lower triangular maps are polyomorphisms, but not necessarily triangular.*

Theorem of van der Kulk & Jung [16,17]. *The group $\mathbf{GA}_2(\mathbb{C})$ of all polyomorphisms of \mathbb{C}^2 is the free product of its two subgroups, the affine group $Af(\mathbb{C}^2)$ and the (lower) triangular group $\mathcal{T}(\mathbb{C}^2)$, amalgamated over their intersection, the group Δ of (lower) triangular affine maps.*

We used this in our **classification of polyflows in 2-dimensions** [18,19].

There are two strange facts about the Theorem of van der Kulk & Jung:

Two Strange Facts. *Knowledge of the group $\mathbf{GA}_2(\mathbb{C})$ has not yet helped: (1) to prove JC in two dimensions; (2) to find the generators for $\mathbf{GA}_3(\mathbb{C})$.*

Yagzhev [20] and Bass, Connell & Wright [21] discovered that given a Keller map $f : \mathbb{C}^m \rightarrow \mathbb{C}^m$ of degree $d \geq 3$, it is possible, *by increasing the number of variables*, to get another map $F : \mathbb{C}^n \rightarrow \mathbb{C}^n$ with $n > m$ of the form $F(x) := x - H(x)$ with polynomial $H(x)$ satisfying $H(tx) \equiv t^3 H(x)$, such that f is injective iff F is injective. Thus, we have the second reduction—

Reduction of Degree to Cubic-Homogeneous Maps. *To prove JC in every dimension, it suffices to prove it (but also in every dimension) for polynomial maps of the form $F(x) := x - H(x)$ with $H(tx) \equiv t^3 H(x)$. For such “cubic-homogeneous” maps, $\det F'(x) \equiv 1$ if and only if $H'(x)^n \equiv 0$.*

This was improved by Drużkowski [22] to give the third reduction:

Reduction to Cubic-Linear Maps. *It suffices to prove JC in each dimension for maps of the form $F_A(x) := x - [\text{diag}(Ax)]^2 Ax$ for a complex matrix A . That is, we may assume that $H(x) = H_A(x) := [\text{diag}(Ax)]^2 Ax$.*

2.2 A restricted weaker form of MYC (RMYC) is equivalent to JC

By these reductions we see that JC is false iff \exists an $n \times n$ complex matrix A such that $H'_A(x) \equiv 3[\text{diag}(Ax)]^2 A$ is nilpotent $\forall x \in \mathbb{C}^n$ and the system

$$\dot{x} = G_A(x) := F_A(x_1) - F_A(x) = c - x + [\text{diag}(Ax)]^2 Ax \quad (\text{RMYC})$$

has two distinct rest points x_1, x_2 . Note that $G'_A(x)$ has all eigenvalues -1 .

The 3D polynomial counterexample to MYC found by Anna Cima et al [43], reproduced below in § 5, is *not of this form!* So RMYC \equiv JC is still open!

3. Progress on The Markus-Yamabe Conjecture 1993–94

We now return to our story of MYC: It is finally proved for class \mathcal{C}^1 vector fields in \mathbb{R}^2 ; and a counterexample is established in \mathbb{R}^4 with a periodic orbit.

3.1 The Markus-Yamabe Conjecture is True in \mathbb{R}^2

In 1993 MYC was proved for class \mathcal{C}^1 vector fields on \mathbb{R}^2 , independently by Robert Feßler [23] at ETH in Zurich, by Carlos Gutierrez [24] at IMPA in Rio de Janeiro, and by A. A. Glutsyuk [25] at Moscow State University. The first two presented their proofs at the Trento Conference [26] organized by Marco Sabatini. The first two proofs are rather long and difficult but prove more general injectivity results than the shorter proof by Glutsyuk.

3.2 Analytic MY-Systems in \mathbb{R}^4 with Periodic Orbits

In 1994, Bernat & Llibre [27], spurred by Barabanov's paper [13], were able to construct an *analytic* MY-SYSTEM in \mathbb{R}^4 that has a nonconstant periodic solution. Consequently, this provides a counterexample to MYC in \mathbb{R}^4 .

EXAMPLE 3. A 4D analytic MY-SYSTEM with a periodic orbit:

$$\left. \begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_4 \\ \dot{x}_3 &= x_1 - 2x_4 - \frac{9131}{900}\psi(x_4) \\ \dot{x}_4 &= x_1 + x_3 - x_4 - \frac{1837}{180}\psi(x_4) \end{aligned} \right\} \quad (3)$$

Bernat & Llibre [27] prove there exists a \mathcal{C}^1 function ψ such that system (3) is a MY-SYSTEM with a (nonconstant) *periodic* orbit. They remark that ψ can be chosen class \mathcal{C}^r , for any $r \geq 1$, or class \mathcal{C}^∞ , or even *analytic*.

Since we have no *polynomial* MY-SYSTEM with a nonconstant *periodic* orbit, this example remains interesting. Is it true that *every forward-bounded solution of a polynomial MY-SYSTEM tends to the rest point $f(0) = 0$* ?

4. Back to Keller's Jacobian Conjecture

We can now describe our new attempt [29,34] to prove JC which quickly led others to discover, serendipitously, how to construct *polynomial* counterexamples to the Markus-Yamabe Conjecture;—first in \mathbb{R}^4 and then in \mathbb{R}^3 . These *polynomial* examples were found after seeing Bo Deng's Lemma [38].

4.1 Can JC be proved by Conjugations $h_s(sF(x)) = sh_s(x)$? [29,34]

Some say “no”,—others say “maybe”. This question is still open, but let me begin at the beginning. Gaetano Zampieri spent the academic year 1993–94 at the University of Nebraska-Lincoln. At lunch on Thursday, March 3rd, 1994, Bo Deng and Gaetano sprung the following bold idea on me: Might it not be possible to prove Keller’s Jacobian Conjecture by means of a conjugation h_s depending on a complex parameter s ? That is, given a polynomial mapping $F : \mathbb{C}^n \rightarrow \mathbb{C}^n$ satisfying Keller’s condition $\det F'(x) \equiv 1$, could it be proved that, for at least one complex value of s other than 0 or 1, there exists an injective map $h_s : \mathbb{C}^n \rightarrow \mathbb{C}^n$ such that

$$h_s(sF(x)) = sh_s(x) \text{ for all } x \text{ in } \mathbb{C}^n? \quad (4)$$

It would follow immediately from this that F is also injective, thus proving Keller’s Conjecture! Bo Deng reminded me that this might have some hope because Poincaré had proved in his thesis that an *analytic* map can be formally conjugated *locally* to its linear part in the neighborhood of a fixed point; and later (c. 1942) C. L. Siegel [28] had studied the local convergence of Poincaré’s formal power series for the conjugation function $h(x)$. This theory of **non-resonant linearization** shows that dilated polymorphisms $sF(x) = sx - sH(x)$ are *analytically* conjugate to sx , at least *locally*, if $|s| \neq 1$. To my surprise, this conjugation function $h_s(x)$ turned out to be *global* and *polynomial in x* , for *all* cubic-homogeneous examples of F that I worked out in March 1994 and presented at Curaçao [29] in July 1994.

4.2 Resonance and Poincaré's Theorem

We take the following results on Poincaré's theory of **resonance** and Siegel's theorem on **multiplicity-type** (C, ν) from Chapter 5 of Arnold's book [30].

Resonance of order $|m|$. The n -tuple $\sigma = (s_1, \dots, s_n)$ of eigenvalues of a linear map L , such as the derivative map $x \mapsto F'(0)x$ at the fixed point $x = 0$ of a mapping $F(x)$, is said to be **resonant of order $|m|$** if one of these eigenvalues (say s_k) satisfies: $s_k = \sigma^m = s_1^{m_1} \cdots s_n^{m_n}$, for integers $m_j \geq 0$ with $|m| = m_1 + \cdots + m_n \geq 2$.

Poincaré's Theorem. Let F be analytic at $x = 0$, $F(0) = 0$, and assume the eigenvalues $\sigma = (s_1, \dots, s_n)$ of $F'(0)$ are *not resonant* (of any order); then there exists a formal power series $h(x)$ so that

$$h \circ F \circ h^{-1}(x) = F'(0)x.$$

4.3 Siegel's Theorem and Dilated Polyomorphisms

Siegel's Theorem. Let F be analytic at $x = 0$ with $F(0) = 0$. If the eigenvalues $\sigma = (s_1, \dots, s_n)$ of $F'(0)$ are of **multiplicity-type** (C, ν) , for constants $C > 0$ and $\nu > 0$, in the sense that

$$|s_j - \sigma^m| \geq C/|m|^\nu \tag{5}$$

$\forall m = (m_1, \dots, m_n)$ with $|m| \geq 2$ and $1 \leq j \leq n$, then the formal power series for h and h^{-1} which occur in Poincaré's Theorem *converge* near $x = 0$.

The Case of Dilated Polyomorphisms. For each complex s , the eigenvalues of the linear map $x \mapsto sF'(0)$ are all equal to s . So they are resonant

of order $|m| \geq 2$ if and only if $s = s^{m_1} \cdots s^{m_n} = s^{|m|}$, or $s^{|m|-1} = 1$, or s is a root of unity. On the other hand, if $|s| \neq 0$ or 1 , then Siegel's inequality (5) holds for $C = ||s| - |s|^2|$ and all $\nu \geq 1$, because $\sigma^m = s^{|m|}$ so that

$$|s - s^{|m|}| \geq ||s| - |s|^{|m|}| \geq ||s| - |s|^2| > C/2 \geq C/|m|^\nu.$$

4.4 Polynomial Conjugations of Dilated Polyomorphisms

Here is a sample of what I presented at Curaçao [29] on July 4th, 1994. The question raised there is this: Does the Poincaré-Siegel method of *local linearization* for *analytic* maps turn out to be *global linearization* for *polynomial maps*? Does this always happen for the special cubic-homogeneous maps; or at least for the even more special cubic-linear maps? Just exactly when does h_s turn out to be a *global linearization*? Between March 3rd and July 4th, 1994, I had worked out the details of the situation for about 24 polynomial mappings F , all of which are polyomorphisms, and most of them either cubic-homogeneous or cubic-linear. In all these cases the corresponding conjugation function h_s (called **the Schröder function for F**) turned out to be itself a polyomorphism in x for each complex number s for which $|s| \neq 1$. In fact, each h_s I computed at that time turned out to be polynomial in x with coefficients rational functions of s whose denominators are zero only at certain roots of unity. Here below are two examples: A cubic-homogeneous map (that is not cubic-linear), and a cubic-linear map; each with their associated Schröder maps h_s and h_s^{-1} .

EXAMPLE 4. Anick's non-triangularizable cubic-homogeneous polyomorphism has been studied by M. K. Smith [31], D. Wright [32], and E.-M.G.M. Hubbers [33]. It is not known to be **tame**: i.e., in the subgroup of $\mathbf{GA}_n(\mathbb{C})$ generated by the triangular and affine polyomorphisms of \mathbb{C}^n . But it was shown to be **stably tame** [31]: i.e., for some choice of additional variables x_{n+1}, \dots, x_m , the extension of F to \mathbb{C}^m which fixes x_{n+1}, \dots, x_m is tame. By the Theorem of van der Kulk & Jung, all 2-dimensional polyomorphisms are tame. For $n \geq 3$ it's likely that nontame polyomorphisms exist; but that's an open question. Here is Anick's polyomorphism of \mathbb{C}^4 :

$$f(x) = \begin{bmatrix} x_1 \\ x_2 \\ x_3 - x_2(x_1 x_3 + x_2 x_4) \\ x_4 + x_1(x_1 x_3 + x_2 x_4) \end{bmatrix} \quad f^{-1}(y) = \begin{bmatrix} y_1 \\ y_2 \\ y_3 + y_2(y_1 y_3 + y_2 y_4) \\ y_4 - y_1(y_1 y_3 + y_2 y_4) \end{bmatrix}$$

And here are its Schröder functions h_s and h_s^{-1} that I presented at Curaçao:

$$h_s(x) = \begin{bmatrix} x_1 \\ x_2 \\ x_3 + x_2(x_1 x_3 + x_2 x_4)/(s^2 - 1) \\ x_4 - x_1(x_1 x_3 + x_2 x_4)/(s^2 - 1) \end{bmatrix}$$

$$h_s^{-1}(y) = \begin{bmatrix} y_1 \\ y_2 \\ y_3 - y_2(y_1 y_3 + y_2 y_4)/(s^2 - 1) \\ y_4 + y_1(y_1 y_3 + y_2 y_4)/(s^2 - 1) \end{bmatrix}$$

EXAMPLE 5. A cubic-linear polyomorphism $F_A(x) := x - [\text{diag}(Ax)]^2 Ax$:

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad F_A = \begin{pmatrix} x - y^3 \\ y - z^3 \\ z \end{pmatrix}$$

$$F_A^{-1} = \begin{pmatrix} x + y^3 + 3y^2 z^3 + 3y z^6 + z^9 \\ y + z^3 \\ z \end{pmatrix}$$

—and here is what I found for its *polynomial* Schröder map $h_s =$

$$\left(\begin{array}{c} x + \frac{y^3}{s^2-1} + \frac{3s^2 y^2 z^3}{(s^2-1)(s^4-1)} + \frac{3s^2 (s^4+1) y z^6}{(s^2-1)(s^4-1)(s^6-1)} + \frac{s^2 (1-s^2+3s^4-s^6+s^8) z^9}{(s^2-1)^2 (s^6-1)(s^8-1)} \\ y + \frac{z^3}{s^2-1} \\ z \end{array} \right)$$

and $h_s^{-1} =$

$$\left(\begin{array}{c} x + \frac{y^3}{s^2-1} + \frac{3y^2 z^3}{(s^2-1)(s^4-1)} - \frac{3(s^2+1) y z^6}{(s^2-1)(s^4-1)(s^6-1)} + \frac{z^9}{(s^2-1)^3 (s^8-1)} \\ y - \frac{z^3}{s^2-1} \\ z \end{array} \right)$$

4.5 Cubic-homogeneous $F = I - H$ whose h_s is not polynomial

Meanwhile, Bo Deng and Gaetano Zampieri wrote up our failed attempt [34] to prove directly that *the Schröder function h_s of a dilated Keller map sF is a holomorphic automorphism* which thus conjugates sF to sI globally. This has been called the DMZ-CONJECTURE. At Curaçao I offered a prize of \$100 to the first person to find an example of a *cubic-homogeneous polymorphism* whose Schröder function h_s is *not polynomial*. On September 16 & 19, 1994, I received e-mail messages from Arno van den Essen <essen@sci.kun.nl> in Nijmegen, The Netherlands. The first message contained such an example in six variables; and the second message contained a simpler example in four variables. This second example appeared in October 1994 as a Nijmegen University Report [35], and as a note at the end of the Curaçao Proceedings under “Reactions on the conference” [36]. On Tuesday, October 4, 1994, I sent Arno van den Essen a bank draft for \$100. Here is Arno’s second example [36]—the one I received on September 19, 1994.

EXAMPLE 6. The \$100 cubic-homogeneous F whose h_s is not polynomial.

Let $p(x) = x_1x_3 + x_2x_4$. Then Arno's map $F(x) = x - H(x)$ is:

$$F(x_1, x_2, x_3, x_4) = (x_1 + p(x)x_4, x_2 - p(x)x_3, x_3 + x_4^3, x_4). \quad (6)$$

Of course, higher-dimensional examples are made by forming the direct sum of this example with the identity in the additional variables. Although sF is *not polynomially conjugate* to its linear part sx , it is *globally analytically conjugate* to sx [37]. So at this point the DMZ-CONJECTURE made in [34] was still open. What was needed was a *criterion* to help find maps F whose dilations are *not globally analytically conjugate* to their linear part. Bo Deng found such a criterion [38] in the Spring of 1995. As we learned later, it had already been found seven years earlier by Rosay & Rudin [39] in a different context. This criterion is described in the next section.

4.6 Characterization of Globally Linearizable Analytic Maps

In trying to understand why some polyomorphisms sF may not be globally analytically conjugate to their linear parts sx , Bo Deng discovered, in the spring of 1995, the following elegant necessary and sufficient condition [38]:

Bo's Lemma. *A dilation λf of an analytic map f of \mathbb{C}^n into itself, with $f(0) = 0$, $f'(0) = I$, and $|\lambda| > 1$, has a global analytic conjugation to its linear part λx if and only if f is a holomorphism of \mathbb{C}^n and $x = 0$ is a global attractor of the iterates of the inverse of λf .*

More Generally. *If F is an analytic map of \mathbb{C}^n into itself, $F(0) = 0$, $\det F'(0) \neq 0$, and the eigenvalues λ of $F'(0)$ are **not resonant** and satisfy $|\lambda| < 1$; then F is analytically conjugate to $F'(0)$ if and only if F is a holomorphism of \mathbb{C}^n and $x = 0$ is a **global attractor**: i.e., $\forall x \in \mathbb{C}^n$, $F^k(x) := F \circ F^{k-1}(x) \rightarrow 0$, as $k \rightarrow \infty$.*

Franc Forstneric kindly brought to our attention via e-mail on January 10, 1996, that this result is in the 1988 paper [39] by Rosay & Rudin.

But Bo had sent preliminary copies of his preprint [38] to several people including Arno van den Essen in Nijmegen. As we shall see in the next two sections, van den Essen (with help from his student Hubbers and their colleagues in Barcelona) found Bo's Lemma to be just the tool they needed: First to construct a counterexample to the DMZ-CONJECTURE; and second, after some additional searching, to find a *polynomial* counterexample to MYC in \mathbb{R}^3 . So even though Bo's result was not new in the literature, it was Bo's preprint [38] that led van den Essen and his colleagues to find counterexamples to the DMZ-CONJECTURE and to the MY-CONJECTURE.

4.7 There are degree 5 maps $F = I - H$ whose h_s are not analytic

The example given in this section shows that the DMZ-CONJECTURE is false!

EXAMPLE 7. Counterexample to the DMZ-CONJECTURE [40]:

A 4-dimensional, degree 5, polyomorphism F whose complex s -dilations sF are not globally holomorphically conjugate to sx .

**This counterexample was found with the help of Bo's Lemma
by van den Essen & Hubbers in November 1995:**

Define $p(x) = x_1x_3 + x_2x_4$ and let m be any odd integer ≥ 3 . Then

$$F(x) = \left(x_1 + p(x)^2x_4, x_2 - p(x)^2x_3, x_3 + x_4^m, x_4 \right) \quad (7)$$

is a counterexample to the DMZ-CONJECTURE. The authors show that this same example is also a counterexample to the **discrete Markus-Yamabe Question** stated in La Salle's 1976 CBMS-NSF lectures [41, pages 20–21]. This question was rediscovered in 1995 by Cima, Gasull, and Mañosas [42].

5. Polynomial MY-Systems in \mathbb{R}^3 with Unbounded Orbits

By modifying the example in the last section, a 3-dimensional polynomial counterexample to MYC was found [43] by Cima, van den Essen, Gasull, Hubbers & Mañosas just before December, 1995: Namely—

$$\text{(MYC-FALSE)} \begin{cases} \dot{x}_1 = -x_1 + x_3(x_1 + x_2x_3)^2 \\ \dot{x}_2 = -x_2 - (x_1 + x_2x_3)^2 \\ \dot{x}_3 = -x_3 \end{cases} \quad (8)$$

The origin $(0, 0, 0)$ is a rest point, and the Jacobian is

$$f'(x) = \begin{pmatrix} -1 + 2dx_3 & 2dx_3^2 & d^2 + 2dx_2x_3 \\ -2d & -1 - 2dx_3 & -2dx_2 \\ 0 & 0 & -1 \end{pmatrix} \quad (9)$$

where $d = (x_1 + x_2x_3)$. The characteristic equation of $f'(x)$ is

$$\lambda^3 + 3\lambda^2 + 3\lambda + 1 = (\lambda + 1)^3 = 0,$$

so for each x in \mathbb{R}^3 all eigenvalues of $f'(x)$ are -1 .

But (8) has the unbounded solution: $(x_1, x_2, x_3) = (18e^t, -12e^{2t}, e^{-t})$.

If we denote the initial conditions by $c = (c_1, c_2, c_3) = (x_1(0), x_2(0), x_3(0))$,

then, when $c_3 = 0$, all solutions lie in the x_1x_2 -plane and must therefore tend

to 0 as t tends to $+\infty$, because we know that MYC is true in dimension 2:

$$\begin{aligned} x_1(t, c_1, c_2, 0) &= c_1 e^{-t} \\ x_2(t, c_1, c_2, 0) &= (c_2 - c_1^2) e^{-t} + c_1^2 e^{-2t} \\ x_3(t, c_1, c_2, 0) &= 0 \end{aligned} \tag{10}$$

6. Remaining Open Questions for cubic-linear maps $F_A(x)$

Q1. Does there exist, for $0 < |s| \neq 1$, an injective map h_s satisfying (4)

(a) if F_A is a polyomorphism?

(b) if $\det F'_A(x) \equiv 1$; or, equivalently, if $([\text{diag}(Ax)]^2 A)^n \equiv 0$?

Q2. When is $h_s(x)$ global and polynomial? holomorphic? continuous?

Q3. **The cubic-linear linearization conjecture** [29]: Every cubic-linear polyomorphism F_A is *globally analytically conjugate* to its linear part sx ; i.e., its Schröder function $h_s(x)$ is an analytic automorphism of \mathbb{C}^n . Even if this scalar- s version is not true, there is still the matrix version, where the scalar s is replaced by an invertible matrix S with appropriate eigenvalues. It is shown in [45] that the h_s for the 15-dimensional example in [44] is a *polyomorphism*! However, it is also shown in [45] that there do exist cubic-linear maps F_A whose h_s is *not a polyomorphism*, although it is still a *holomorphism*. So question Q3 is still open.

7. Appendix: The Strong Real JC & Mean-Value Formulas

The conditions $\det F'(x) \neq 0$ & $\det F'(x) \equiv 1$ are not equivalent on \mathbb{R}^n .

This gave rise to the following variation of JC called RJC or SRJC:

7.1 The Strong Real Jacobian Conjecture

RJC. *Is every polynomial map $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ with $\det[f'(x)] \neq 0$ on \mathbb{R}^n bijective with an analytic inverse? (Answer: Many are, but not all!)*

EXAMPLE: The map $x \mapsto x + x^3 \in \mathbb{R}^1$ has Jacobian $1 + 3x^2 \geq 1$ and is bijective with analytic, *but not polynomial*, inverse.

Pinčuk's Counterexample to RJC [46]: Define the polynomial map $\langle x, y \rangle \mapsto \langle p, q \rangle$ as follows: $t = xy - 1$, $h = t(xt + 1)$, $f = ((h + 1)/x)(xt + 1)^2$, $u = 170fh + 91h^2 + 195fh^2 + 69h^3 + 75h^3f + (75/4)h^4$, $p(x, y) = f + h$, $q(x, y) = -t^2 - 6th(h + 1) - u$. Degree(p, q) = (10, 25). The Jacobian matrix J of this mapping is a sum of squares so $J \geq 0$; and also we can prove $J \neq 0$:

$$J \equiv \partial(p, q)/\partial(x, y) = t^2 + (t + (13 + 15h)f)^2 + f^2.$$

$J = 0 \Rightarrow t = 0$ & $f = 0$; but then also $h = 0$ & $f = 1/x \neq 0$. But the map is not injective because $\langle p, q \rangle(1, 0) = \langle p, q \rangle(-1, -2) = \langle 0, -1 \rangle$.

The other case, when $\det F'(x) \equiv 1$ on \mathbb{R}^n , is equivalent to JC itself.

7.2 Mean Value Formulas (MVF) & The Bilinear Matrix $\mathcal{B}(x, y)$ [47]

QUADRATIC-MVF. For deg 2 maps F ,
$$\boxed{F(x) - F(y) \equiv F' \left(\frac{x+y}{2} \right) (x-y)}.$$

Therefore, any *quadratic* map F is *injective* if $\det F'(x)$ is never zero.

Theorem. To a polynomial map $H : \mathbb{C}^n \rightarrow \mathbb{C}^n$ satisfying $H(tx) \equiv t^3 H(x)$ corresponds a unique matrix-valued bilinear map $(x, y) \mapsto \mathcal{B}(x, y)$ such that

- (1) $\mathcal{B}(x, x) = H'(x), \quad \forall x \in \mathbb{C}^n,$
- (2) $\mathcal{B}(x, y) = \mathcal{B}(y, x), \quad \forall x, y \in \mathbb{C}^n,$
- (3) $\mathcal{B}(x, y)z = \mathcal{B}(x, z)y, \quad \forall x, y, z \in \mathbb{C}^n.$ Also,
- (4) $H(x) = (1/3)H'(x)x = (1/3)\mathcal{B}(x, x)x.$

Namely, $\mathcal{B}(x, y) := \frac{1}{4}\{H'(x+y) - H'(x-y)\}$. Furthermore:

If we define the mapping $F(x) := x - H(x)$, then $F'(x) = I - H'(x)$, and

$$\boxed{\det F'(x) \equiv 1 \text{ iff } H'(x)^n \equiv 0.}$$

If $H_A(x) := [\text{diag}(Ax)]^3 \mathbf{1}$ is defined by a matrix A , then $F_A(x) = x - H_A(x)$ is **cubic-linear** with bilinear-matrix $\mathcal{B}_A(x, y) := 3[\text{diag}(Ax)][\text{diag}(Ay)]A$.

CUBIC-MVF. For polynomial maps $F(x) := x - H(x)$, with $H(tx) = t^3 H(x)$,

$$\boxed{F(x) - F(y) = [I - (1/3)\{\mathcal{B}(x, x) + \mathcal{B}(x, y) + \mathcal{B}(y, y)\}](x-y).} \quad (\text{MVF3})$$

By the transformation $x = au + \bar{a}v$ and $y = \bar{a}u + av$ in (MVF3), where $a = (1 + \sqrt{3}i)/2$, we obtain $F(x) - F(y) \equiv (i\sqrt{3})[I - \mathcal{B}(u, v)](u - v)$. So $F : \mathbb{C}^n \leftrightarrow$ is *injective* if and only if $\mathcal{B}(x, y)(x - y) = (x - y)$ only if $x = y$.

We call this the *phantom eigenvector formulation* of JC.

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